Contributions in combinatorics in commutative algebra

Ph.D. Thesis

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Abstract

In the first chapter we present new results related on monomial ideals of Borel type. Also, we introduce a new class of monomial ideals, called **d**-fixed ideals, which generalize the class of p-Borel ideals and we extend several results to this new class. In section 1.2 we extend a result of Eisenbud-Reeves-Totaro in the frame of ideals of Borel type. This allows us to obtain some nice consequences related to the regularity of the Borel type ideals. In section 1.3 we introduce a new class of ideals, called strong Borel type ideals, and we compute the Mumford-Castelnouvo regularity for a special case of strong Borel type ideals. In the sections 1.4, 1.5 and 1.6 we show how some results for p-Borel ideals can be transfered to **d**-fixed ideals. In particular, we give the form of a principal **d**-fixed ideal and we compute the socle of factors of these ideals, using methods similar as in [23], see section 1.5. This allows us to give a generalization of Pardue's formula, i.e. a formula of the regularity for a principal **d**-fixed ideal, see section 1.6. In the last section of the first chapter, we describe the **d**-fixed ideals generated by powers of variables.

In the second chapter, we compute the generic initial ideal, with repect to the reverse lexicographic order, of an ideal which define a complete intersection of embedding dimension three with strong Lefschetz property and we show that it is an almost reverse lexicographic ideal, see sections 2.2 and 2.3. This enable us to give a proof for Moreno's conjecture in the case $n = 3$ and characteristic zero, see section 2.1. In section 2.4 we prove that the d-component of the generic initial ideal, with respect to the reverse lexicographic order, of an ideal generated by a regular sequence of homogeneous polynomials of degree d is revlex, in a particular, but important, case. Using this property, in section 2.5 we compute the generic initial ideal for several complete intersections with strong Lefschetz property.

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Contents.

Introduction

In the first chapter, we prove a stable property for monomial ideals of Borel type and give some nice consequences. Also, we discuss issues related to **d**-fixed ideals, a new class of ideals which generalize the class of p-Borel ideals. In order to explain the context, we need some preparations.

Let K be an infinite field, and let $S = K[x_1, ..., x_n], n \geq 2$ the polynomial ring over K. Bayer and Stillman [5] note that a Borel fixed ideal $I \subset S$ satisfies the following property:

$$(*) \quad (I : x_j^\infty) = (I : (x_1, \ldots, x_j)^\infty) \; for \; all \; j = 1, \ldots, n.$$

Herzog, Popescu and Vladoiu [23] say that a monomial ideal I is of *Borel type*, if it satisfy $(*)$. We mention that this concept appears also in [6] as the so called *weakly stable ideal*. Herzog, Popescu and Vladoiu proved in [23] that I is of Borel type, if and only if for any monomial $u \in I$ and for any $1 \leq j < i \leq n$ and $q > 0$ with $x_i^q | u$, there exists an integer $t > 0$ such that $x_j^t u / x_i^q \in I$. In the first section, we present some facts on ideals of Borel type, following [23]. In the second section we prove that if I is an ideal of Borel type, then $I_{\geq e} =$ the ideal generated by the monomials of degree $\geq e$ from I, is stable whenever $e \geq reg(I)$ (Theorem 1.2.10). This allows us to give a generalization of a result of Eisenbud-Reeves-Totaro (Corollary 1.2.11). Also, we prove that the regularity of a product of ideals of Borel type is bounded by the sum of the regularity of those ideals (Theorem 1.2.15). In the third section, we introduce a new class of ideals, called ideals of strong Borel type and we compute the regularity of a principal strong Borel type ideal in a special case (1.3.6).

A p-Borel ideal is a monomial ideal which satisfies certain combinatorial condition, where $p > 0$ is a prime number. It is well known that any positive integer a has an unique p-adic decomposition $a = \sum_{i \geq 0} a_i p^i$. If a, b are two positive integers, we write $a \leq_p b$ iff $a_i \leq b_i$ for any i, where $a = \sum_{i \geq 0} a_i p^i$ and $b = \sum_{i \geq 0} b_i p^i$. We say that a monomial ideal $I \subset S = K[x_1, \ldots, x_n]$ is p-Borel if for any monomial $u \in I$ and for any indices $j < i$, if $t \leq_p \nu_i(u)$ then $x_j^t u / x_i^t \in I$, where $\nu_i(u) = max\{k : \; x_i^k | u\}$.

This definition suggests a natural generalization. The idea is to consider a strictly increasing sequence of positive integers $\mathbf{d} : 1 = d_0 | d_1 | \cdots | d_s$, which we called a **d**-sequence. Lemma 1.4.1 states that for any positive integer a, there exists an unique decomposition $a = \sum_{i=0}^{s} a_i d_i$ with $0 \leq a_i < d_{i+1}/d_i$ for any i. If a, b are two positive integers, we write $a \leq_{\mathbf{d}} b$ iff $a_i \leq b_i$ for any i, where $a = \sum_{i=0} a_i d_i$ and $b = \sum_{i=0} b_i d_i$. We say that a monomial ideal $I \subset S$ is **d**-fixed if for any monomial $u \in I$ and for any indices $j < i$, if $t \leq_{\mathbf{d}} \nu_i(u)$ then $x_j^t u / x_i^t \in I$. Obviously, the p-Borel ideals are a special case of **d**-fixed ideals for $\mathbf{d} : 1 | p | p^2 | \cdots$.

A principal **d**-fixed ideal, is the smallest **d**-fixed ideal which contains a given monomial. 1.4.8 and 1.4.11 give the explicit form of a principal **d**-fixed ideal. In section 1.5 we compute the socle of factors for a principal **d**-fixed ideal (1.5.1 and 1.5.4). The proofs are similar as in [23] but we consider that is necessary to present them in this context.

In the section 1.6 we give a formula (Theorem 1.6.1) for the regularity of a principal **d**-fixed ideal I, which generalize the Pardue's formula for the regularity of principal p-Borel ideals, proved by Aramova-Herzog [3] and Herzog-Popescu [24]. Using a theorem of Popescu [29] we compute the extremal Betti numbers of S/I (1.6.3). S.Ahmad and I.Anwar proved in [1] than any ideal of Borel type has the regularity bounded by the number $q(I) = m(I)(deg(I) - 1) + 1$, where $m(I) = max\{i : \ x_i|u \text{ for some } u \in G(I)\}$, $deg(I) = max\{deg(u) : \ u \in G(I)\}$ and $G(I)$ is the set of minimal monomial generators of I. As a consequence of Pardue's formula, we obtain another proof of this result in the particular case of **d**-fixed ideals (Corollary 1.6.2). Also, we introduce a new class of monomial ideals, called $\mathcal{D}$-fixed ideals, which are sums of various **d**-fixed ideals . Since any **d**-fixed ideal is in particular, an ideal of Borel type, it follows that Theorem 1.2.10 can be applied for them, so we get Corollary 1.6.6. This result was first obtain by Herzog-Popescu [24] in the special case of a principal p-Borel ideal. In the last section we give the explicit form of a **d**-fixed ideal generated by powers of variables (Proposition 1.7.2) and make some remarks on its regularity.

In the second chapter, we discuss issues related to the generic initial ideal of an ideal generated by a regular sequence of homogeneous polynomials.

Let $S = K[x_1, x_2, x_3]$ be the polynomial ring over a field K of characteristic zero. Let f_1, f_2, f_3 be a regular sequence of homogeneous polynomials of degrees d_1, d_2 and d_3 respectively. We consider the ideal $I = (f_1, f_2, f_3) \subset S$. Obviously, S/I is a complete intersection Artinian K-algebra. One can easily check that the Hilbert series of S/I depends only on the numbers d_1, d_2 and d_3. More precisely,

$$H(S/I, t) = (1 + t + \cdots + t^{d_1-1})(1 + t + \cdots + t^{d_2-1})(1 + t + \cdots + t^{d_3-1}).$$

[23, Lemma 2.9] gives an explicit form of $H(S/I, t)$.

Let $S = K[x_1, \ldots, x_n]$ and let $I = (f_1, \ldots, f_n) \subset S$ be an ideal generated by a regular sequence of homogeneous polynomials. We say that a homogeneous polynomial f of degree d is *semiregular* for S/I if the maps $(S/I)_t \xrightarrow{\cdot f} (S/I)_{t+d}$ are either injective, either surjective for all $t \geq 0$. We say that S/I has the *weak Lefschetz property* (WLP) if there exists a linear form $\ell \in S$, semiregular on S/I, in which case we say that ℓ is a weak Lefschetz element for S/I. A theorem of Harima-Migliore-Nagel-Watanabe (see [20]) states that S/I has (WLP) for $n = 3$. We say that S/I has the *strong Lefschetz property* (SLP) if there exists a linear form $\ell \in S$ such that ℓ^b is semiregular on S/I for all integer $b \geq 1$. In this case, we say that ℓ is a strong Lefschetz element for S/I. Of course, $(SLP) \Rightarrow (WLP)$ but the converse is not true in general. In the case $n = 3$, it is not known if S/I has (SLP) for any regular sequence of homogeneous polynomials f_1, f_2, f_3. However, this is known true for certain cases, for example, when f_1, f_2, f_3 is generic, see [27] or when $f_2 \in K[x_2, x_3]$ and $f_3 \in K[x_3]$, see [21] and [22].

We say that a property (P) holds for a *generic* sequence of homogeneous polynomials $f_1, f_2, \ldots, f_n \in S = K[x_1, x_2, \ldots, x_n]$ of given degrees $d_1, d_2, \ldots, d_n$ if there exists a nonempty open Zariski subset $U \subset S_{d_1} \times S_{d_2} \times \cdots \times S_{d_n}$ such that for every $(f_1, f_2, \ldots, f_n) \in U$ the property (P) holds. For example, a generic sequence of homogeneous polynomials $f_1, f_2, \ldots, f_n \in S$ is regular.

Now, we present some conjectures and the relations between them (see [27]).

Conjecture A.(Fröberg) If $f_1, f_2, \ldots, f_r \in S = K[x_1, \ldots, x_n]$ is a generic sequence of homogeneous polynomials of given degrees $d_1, d_2, \ldots, d_r$ and $I = (f_1, f_2, \ldots, f_r)$ then the Hilbert series of S/I is

$$H(S/I) = \left| \frac{\prod_{i=1}^{r}(1-t^{d_i})}{(1-t)^n} \right|,$$

where $|\sum_{j \geq 0} a_t t^j| = \sum_{j\geq 0} b_t t^j$, with $b_j = a_j$ if $a_i > 0$ for all $i \leq j$ and $b_j = 0$ otherwise.

Conjecture B. If $f_1, f_2, \ldots, f_n \in S = K[x_1, \ldots, x_n]$ is a generic sequence of homogeneous polynomials of given degrees $d_1, d_2, \ldots, d_n$ and $I = (f_1, \ldots, f_n)$ then $x_n, x_{n-1}, \ldots, x_1$ is a semi-regular sequence on $A = S/I$, i.e. x_i is semiregular on $A/(x_n, \ldots, x_{i+1})$ for all $1 \leq i \leq n$.

Conjecture C. If $f_1, f_2, \ldots, f_n \in S = K[x_1, \ldots, x_n]$ is a generic sequence of homogeneous polynomials of given degrees $d_1, d_2, \ldots, d_n$, $I = (f_1, \ldots, f_n)$ and J is the initial ideal of I with respect to the revlex order, then $x_n, x_{n-1}, \ldots, x_1$ is a semi-regular sequence on $A = S/(f_1, \ldots, f_n)$.

Conjecture D.(Moreno) If $f_1, f_2, \ldots, f_n \in S = K[x_1, \ldots, x_n]$ is a generic sequence of homogeneous polynomials of given degrees $d_1, d_2, \ldots, d_n$, $I = (f_1, \ldots, f_n)$ and J is the initial ideal of I with respect to the revlex order, then J is an almost revlex ideal, i.e. if $u \in J$ is a minimal generator of J then every monomial of the same degree which preceeds u must be in J as well.

Pardue proved in [27] that if conjecture A is true for some positive integer n then the conjecture B is true for the same n. Also, conjecture C is true for n if and only if B is true for n and if conjecture B is true for some r then A is true for $n < r$ and exactly for that r. Also, if conjecture D is true for some n then B, and thus C, are true for the same n. Fröberg [19] and Anick [2] proved that A is true for $n \leq 3$ and so B and C are true for $n \leq 3$. Moreno [26] remarked that D is true for $n = 2$. Note that Conjecture A for $n = 3$ does not imply the Moreno's conjecture D for $n = 3$.

Let $I \subset S = K[x_1, \ldots, x_n]$ be a graded ideal, i.e. an ideal generated by homogeneous polynomials. We choose a monomial order "$\leq$" on the set of monomials of S. If $\alpha = (\alpha_{ij})$ is a $n \times n$ invertible matrix with entries in K and $f \in I$ is a polynomial, we denote by αf the polynomial obtained from f by the changing of variables, $x_i \mapsto \sum_{j=1}^{n} \alpha_{ij} x_j$ for all $i = 1, \ldots, n$. We denote $\alpha I = (\alpha f |\ f \in I)$. Galligo and Bayer-Stillman proved that there exists a nonempty open Zariski subset $U \in GL_n(K)$, such that for any $\alpha, \alpha' \in U$, $in_{\leq}(\alpha I) = in_{\leq}(\alpha' I)$. For an $\alpha \in U$, we denote $in_{\leq}(\alpha I) := gin_{\leq}(I)$ and we called it the generic initial ideal of I, with respect to "$\leq$". For an introduction on generic initial ideal,

see [18, §15.9]. The generic initial ideal is Borel fixed. In the case of characteristic zero, that means that it is strongly stable and in the case of positive characteristic p that means it is p-Borel. This remark shows a connection between the two parts of my thesis.

Now, let $I = (f_1, f_2, f_3) \subset S = K[x_1, x_2, x_3]$ an ideal generated by a regular sequence of homogeneous polynomials f_1, f_2, f_3 of degrees d_1,d_2 and d_3, respectively. Let $J = Gin(I)$ be the generic initial ideal of I, with respect to the reverse lexicographic order. Our aim is to compute J for all regular sequences f_1, f_2, f_3 of homogeneous polynomials of given degrees d_1, d_2, d_3 such that S/I has (SLP). We will do this in the sections 2.2 and 2.3. These computations shown us in particular, that J depends only on the numbers d_1, d_2, d_3 (this has been proved also by Popescu and Vladoiu in [31]) and more important, that J is an almost reverse lexicographic ideal (Theorem 2.1.1). As a consequence, conjecture Moreno (D) is true for $n = 3$ and $char(K) = 0$ (Theorem 2.1.2).

Now, let K be an algebraically closed field of characteristic zero. Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over K. Let $n, d \geq 2$ be two integers. We consider $I = (f_1, \ldots, f_n) \subset S$ an ideal generated by a regular sequence $f_1, \ldots, f_n \in S$ of homogeneous polynomials of degree d. We say that $A = S/I$ is a (n, d)-*complete intersection*. Let $J = Gin(I)$ be the generic initial of I, with respect to the reverse lexicographic (revlex) order. With the above notations, Conca and Sidman [6] proved that $J_d =$ the set of monomials of degree d of J, is revlex if $f_1, \ldots, f_n$ is a generic regular sequence, (see [6, Theorem 1.2]).

In the section 2.4, we prove that J_d is a revlex set in another case, namely, when $f_i \in K[x_i, \ldots, x_n]$ for all $1 \leq i \leq n$. It is likely to be true that J_d is revlex for any (n, d)-complete intersection, but we do not have the means to prove this assertion. As Example 2.4.10 shows, the hypotheses $char(K) = 0$ and $f_1, \ldots, f_n$ is a regular sequence are essential.

In the section 2.5, we compute the generic initial ideal for some particular cases of (n, d)-complete intersections: $(n = 4, d = 2)$, $(n = 5, d = 2)$ and $(n = 4, d = 3)$. In order to do this, we suppose in addition that S/I has (SLP). Note that this property holds for generic complete intersection (see [27]) and for the case when $f_i \in k[x_i, \ldots, x_n]$. Also, it was conjectured that (SLP) holds for any standard complete intersection. By a theorem of Wiebe [34], S/I has (WLP) (respectively (SLP)) if and only if x_n is a weak (respectively strong) Lefschetz element for S/J. This result is very important for our computations.

In the writing on this thesis, we used new results from our articles and preprints. In the section 1.2 we followed [10] and [13]. In the sections 1.4, 1.5 and 1.6 we followed [7] and [8]. In the sections 1.3 and 1.7 we used [12]. In the sections 2.1, 2.2 and 2.3 we followed [9] and in the sections 2.4 and 2.5 we followed [11], respectively.

Chapter 1

Ideals of Borel type and d-fixed ideals.

1.1 Ideals of Borel type.

Let K be a field and $S = K[x_1, \ldots, x_n]$ the ring of polynomials over K. Herzog, Popescu and Vlădoiu introduced in [23] the following definition.

Definition 1.1.1. *A monomial ideal $I \subset k[x_1, \ldots, x_n]$ is said to be of Borel type if*

$$(I : x_j^\infty) = (I : (x_1, \ldots, x_j)^\infty),\ for\ any\ j = 1, \ldots, n.$$

We have the following equivalent characterization of ideals of Borel type.

Proposition 1.1.2. *[23, Proposition 2.2] Let $I \subset S$ be a monomial ideal. The following conditions are equivalent:*

(a) I is an ideal of Borel type.

(b) For any $1 \leq j < i \leq n$, we have $(I : x_i^\infty) \subset (I : x_j^\infty)$;

(c) Let $u \in I$ be a monomial and suppose that $x_i^q | u$ for some $q > 0$. Then for any $j < i$ there exists an integer t such that $x_j^t u / x_i^q \in I$;

(d) Let $u \in I$ be a monomial; then for any $1 \leq j < i \leq n$, there exists an integer $t > 0$ such that $x_j^t u / x_i^{\nu_i(u)} \in I$.

Moreover, it is easy to see that the conditions (c) and (d) are satisfied if and only if they are satisfied for all $u \in G(I)$.

Proof. $(a) \Rightarrow (b)$ is trivial. For the converse, we use induction on $1 \leq j \leq n$, the assertion being obvious for $j = 1$. Suppose $j < n$ and $(I : x_j^\infty) = (I : (x_1, \ldots, x_j)^\infty)$. Since by (b), $(I : x_{j+1}^\infty) \subset (I : x_j^\infty)$ it follows that $(I : x_{j+1}^\infty) \subset (I : (x_1, \ldots, x_j)^\infty)$ and thus $(I : x_{j+1}^\infty) \subset (I : (x_1, \ldots, x_{j+1})^\infty)$. Since the converse inclusion it is obvious, we get the required conclusion.

$(c) \Rightarrow (d)$ is trivial. For the converse, let $u \in I$ be a monomial such that $x_i^q | u$ for some $q > 0$ and let $j < i$. By (d) there exists t such that $x_j^t u / x_i^{\nu_i(u)} \in I$. Therefore $x_j^t u / x_i^q = x_i^{\nu_i(u) - q} x_j^t u / x_i^{\nu_i(u)} \in I$.

$(b) \Rightarrow (c)$: Let $u \in I$ be a monomial such that $x_i^q | u$ for some $q > 0$ and let $j < i$. Then $u = x_i^q v$ with $v \in (I : x_i^\infty)$. Therefore, there exists t such that $x_j^t u/x_i^q = x_j^t v \in I$.

$(c) \Rightarrow (b)$: Let $u \in (I : x_i^\infty)$ be a monomial. Then $x_i^q u \in I$ for some $q > 0$ and so (c) implies that $x_j^t u \in I$ for some t, that is, $u \in (I : x_j^\infty)$. □

Proposition 1.1.3. *(a) If $I, J \subset S$ are two ideals of Borel type then $I + J$, $I \cap J$ and $I \cdot J$ are also ideals of Borel type.*

(b) If $I \subset S$ is an ideal of Borel type and $J \subset S$ is an arbitrary monomial ideal, then $(I : J)$ is an ideal of Borel type.

Proof. (a) Since a monomial of $I + J$ is either in I, either in J it follows immediately that $I + J$ is of Borel type, using the characterization (d) from the previous proposition. A similar argument holds for $I \cap J$. Now, let $u \in I \cdot J$ be a monomial. It follows that $u = v \cdot w$, where $v \in I$ and $w \in J$ are monomials. Let $1 \leq i \leq n$ such that $x_i | u$ and let $1 \leq j < i$. Since I is of Borel type, then there exists some $t_1 \geq 0$ such that $x_j^{t_1} \cdot v/x_i^{\nu_i(v)} \in I$. Analogously, there exists some $t_2 \geq 0$ such that $x_j^{t_2} \cdot w/x_i^{\nu_i(w)} \in J$. It follows that $x_j^{t_1+t_2} u/x_i^{\nu_i(u)} \in I \cdot J$, therefore $I \cdot J$ is of Borel type.

(b) Suppose $J = (v_1, \ldots, v_m)$, where $v_i \in S$ are monomials. Since $(I : J) = \cap_{k=1}^m (I : v_k)$ and the intersection of Borel type ideals is still of Borel type, we can assume $m = 1$. Denote $v_1 := v$. Let $u \in (I : v)$ be a monomial. We have $u \cdot v \in I$. Let $1 \leq i \leq n$ such that $x_i | u$ and let $1 \leq j < i$. Since I is of Borel type, there exists some $t \geq 0$ such that $x_j^t u \cdot v/x_i^{\nu_i(uv)} \in I$. In particular, multiplying by $x_i^{\nu_i(v)}$, it follows that $v \cdot (x_j^t u/x_i^{\nu_i(u)}) \in I$ and thus $x_j^t u/x_i^{\nu_i(u)} \in (I : v)$. In conclusion, $(I : v)$ is of Borel type, as required. □

We recall the following definition of Stanley, see [32].

Definition 1.1.4. *Let $S = k[x_1, \ldots, x_n]$ and let M be a finitely generated graded S-module. The module M is sequentially Cohen-Macaulay if there exists a finite filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$ of M by graded submodules of M such that:*

- *M_i/M_{i-1} are Cohen-Macaulay for any $i = 1, \ldots, r$ and*
- *$dim(M_1/M_0) < dim(M_2/M_1) < \cdots < dim(M_r/M_{r-1})$.*

The above filtration is unique and is called the CM-filtration of M

In particular, if $I \subset S$ is a graded ideal then $R = S/I$ is sequentially Cohen-Macaulay if there exists a chain of ideals $I = I_0 \subset I_1 \subset \cdots \subset I_r = S$ such that I_j/I_{j-1} are Cohen-Macaulay and $dim(I_j/I_{j-1}) < dim(I_{j+1}/I_j)$ for any $j = 1, \ldots, r-1$.

Let $I \subset S$ be a monomial ideal. Recursively we define an ascending chain of monomial ideals as follows: We let $I_0 := I$. Suppose I_ℓ is already defined. If $I_\ell = S$ then the chain ends. Otherwise, let $n_\ell = max\{i : \ x_i | u \text{ for an } u \in G(I_\ell)\}$. We set $I_{\ell+1} := (I_\ell : x_{n_\ell}^\infty)$. It is obvious that $n_\ell > n_{\ell+1}$, and therefore the chain $I_0 \subset I_1 \subset \cdots \subset I_r = S$ is finite and has length $r \leq n$. We call this chain of ideals, the *sequential chain* of I. Note that if I is an Artinian monomial ideal then $r = 1$. The converse is true for ideals of Borel type.

Proposition 1.1.5. *[23, Corollary 2.5] Let I be a monomial ideal of Borel type. Then $R = S/I$ is sequentially Cohen-Macaulay.*

Proof. We may assume $I \neq 0$. Let $I = I_0 \subset I_1 \subset \cdots \subset I_r = S$ be the sequential chain of I. Note that, inductively, we get that any ideal I_ℓ is an ideal of Borel type, since $I_{\ell+1}$ is a quotient of I_ℓ. In particular, $I_{\ell+1} = (I_\ell : (x_1, \ldots, x_{n_\ell})^\infty)$ for all ℓ. Fix an integer $\ell < r$. Let $n_j = m(I_j)$ for all j, then the elements of $G(I_j)$ belong to $K[x_1, \ldots, x_{n_\ell}]$ for all $j \geq \ell$. Let J_ℓ be the ideal generated by $G(I_\ell)$ in $K[x_1, \ldots, x_{n_\ell}]$. Then the saturation $J_\ell^{sat} = (J_\ell : (x_1, \ldots, x_{n_\ell})^\infty)$ is generated by the elements of $G(I_{\ell+1})$. It follows that

$$I_{\ell+1}/I_\ell \cong (J_\ell^{sat}/J_\ell)[x_{n_\ell+1}, \ldots, x_n]$$

is an $(n - n_\ell)-$ dimensional Cohen-Macaulay S-module. □

Let M be a finitely generated graded S-module with the minimal graded free resolution $0 \to F_s \to F_{s-1} \to \cdots \to F_0 \to M \to 0$. Let $Syz_t(M) = Ker(F_t \to F_{t-1})$. The module M is called (r,t)-regular if $Syz_t(M)$ is $(r+t)$-regular in the sense that all generators of F_j for $t \leq j \leq s$ have degrees $\leq j + r$. The t-regularity of M is by definition $(t - reg)(M) = min\{r |\ M\ is\ (r,t)-regular\}$. Obvious $(t-reg)(M) \leq ((t-1)-reg)(M)$. If the equality is strict and $r = (t-reg)(M)$ then (r,t) is called a corner of M and $\beta_{t,r+t}(M)$ is an extremal Betti number of M, where $\beta_{ij} = dim_k Tor_i(k, M)_j$ denotes the ij-th graded Betti number of M. Later, we will use the following result of Popescu:

Theorem 1.1.6. *[29, Theorem 3.2] If $I \subset S$ is a Borel type ideal, then S/I has at most $r+1$-corners among $(n_\ell, s(J_\ell^{sat}/J_\ell))$ and the corresponding extremal Betti numbers are*

$$\beta_{n_\ell, s(J_\ell^{sat}/J_\ell)+n_\ell}(S/I) = dim_k(J_\ell^{sat}/J_\ell)_{s(J_\ell^{sat}/J_\ell)}.$$

1.2 Stable properties of Borel type ideals.

It would be appropriate to recall the definition of the Castelnuovo-Mumford regularity. We refer the reader to [18] for further details on the subject.

Definition 1.2.1. *Let K be an infinite field, and let $S = K[x_1, ..., x_n], n \geq 2$ the polynomial ring over K. Let M be a finitely generated graded S-module. The* Castelnuovo-Mumford regularity *$reg(M)$ of M is*

$$\max_{i,j}\{j - i :\ \beta_{ij}(M) \neq 0\}.$$

If M is an artinian S-module, we denote $s(M) = max\{t :\ M_t \neq 0\}$. Herzog, Popescu and Vlădoiu proved the following formula for the regularity of an ideal of Borel type.

Proposition 1.2.2. *[23, Corollary 2.7] If I is a Borel type ideal, with the notations of the section* 1.1*, we have*

$$reg(I) = max\{s(J_0^{sat}/J_0), \ldots, s(J_{r-1}^{sat}/J_{r-1})\} + 1.$$

If $I \subset S$ is a monomial ideal, we denote $q(I) = m(I)(deg(I) - 1) + 1$, where $m(I)$ is the maximal index of a variable which appear in a monomial from $G(I)$ and $deg(I)$ is the maximal degree of a monomial from $G(I)$. We cite the following characterization of ideals of Borel type, given by S.Ahmad and I.Anwar.

Theorem 1.2.3. *[1, Theorem 2.2] Let $I \subset S$ be a monomial ideal. Then the following statements are equivalent:*

1. *I is an ideal of Borel type.*
2. *Each $P \in Ass(S/I)$ has the form $P = (x_1, x_2, ..., x_r)$ for some $1 \leq r \leq n$.*
3. *$I_{\geq q(I)}$ is stable.*

Remark 1.2.4. Note that the implication (1) $\Rightarrow$ (2) follows immediately from 1.1.3(b), since any prime $P \in Ass(S/I)$ can be written as $P = (I : u)$ for some monomial $u \in S$. Another proof is given in [25, Proposition 5.2].

We recall the following result of Eisenbud-Reeves-Totaro.

Proposition 1.2.5. *[17, Proposition 12] Let I be a monomial ideal with $deg(I) = d$ and let $e \geq d$ such that $I_{\geq e}$ is stable. Then $reg(I) \leq e$.*

Corollary 1.2.6. *[1, Corollary 2.4] If I is of Borel type then $reg(I) \leq m(I)(deg(I)-1)+1$. The same holds for a monomial ideal I with $Ass(S/I)$ totally ordered by inclusion.*

Proof. By the Theorem 1.2.3 we have $I_{\geq q(I)}$ stable. As $q(I) \geq deg(I)$ we get $reg(I) \leq q(I)$ by Proposition 1.2.5. For the second statement, we renumber the variables x_i's such that I satisfies (2) from Theorem 1.2.3 and than apply the first statement. □

Remark 1.2.7. The number $m(I) \cdot (deg(I)-1)+1$ is the best possible linear upper bound for the regularity of a Borel type ideal, I. Indeed, if we consider $I = (x_1^2, x_2^2) \subset K[x_1, x_2]$, we have $reg(I) = 3 = 2 \cdot (deg(I) - 1) + 1$. See also, [1, Remark 1.3].

Lemma 1.2.8. *Let $I \subset S$ be a monomial ideal and $I' = IS'$ the extension of I in $S' = S[x_{n+1}]$. If $e \geq deg(I)$, then $I_{\geq e}$ is stable if and only if $I'_{\geq e}$ is stable.*

Proof. Let $u \in I'_{\geq e}$ be a monomial. Then $u = x_{n+1}^k \cdot v$ for some $v \in I$. If $k > 0$ then $m(u) = n + 1$ and therefore, for any $i < n + 1$, $x_i \cdot u/x_{n+1} = x_{n+1}^{k-1} \cdot x_i \cdot v \in I'_{\geq e}$. If $k = 0$ then $m(u) \leq n$ and since $I_{\geq e}$ is stable, it follows $x_i \cdot u/x_{m(u)} \in I'_{\geq e}$. Thus $I'_{\geq e}$ is stable. For the converse, simply notice that $G(I_{\geq e}) \subset G(I'_{\geq e})$ and since is enough to check the stable property only for the minimal generators, we are done. □

Lemma 1.2.9. *If $I \subset S$ is an Artinian monomial ideal and $e \geq reg(I)$ then $I_{\geq e}$ is stable.*

Proof. Since I is Artinian, it follows that the length of the sequential chain of I is $r = 1$. By 1.2.2 we get $reg(I) = s(S/I) + 1$ and therefore, if $e \geq reg(I)$ then $I_{\geq e} = S_{\geq e}$, thus $I_{\geq e}$ is stable. □

Theorem 1.2.10. *Let $I \subset S$ be a Borel type ideal and let $e \geq reg(I)$ be an integer. Then $I_{\geq e}$ is stable.*

Proof. We use induction on $r \geq 1$, where r is the length of the sequential chain of I. If $r = 1$, i.e. I is an Artinian ideal, we are done as in the proof of the previous lemma.

Suppose now $r > 1$ and let $I = I_0 \subset I_1 \subset \cdots \subset I_r = S$ be the sequential chain of I. Let $n_\ell = m(I_\ell)$ for $0 \leq \ell \leq r$. Let $J_\ell \subset S_\ell = K[x_1, \ldots, x_{n_\ell}]$ be the ideal generated by $G(I_\ell)$. Using the induction hypothesis, we may assume $(I_1)_{\geq e}$ stable for $e \geq reg(I_1)$. On the other hand, from 1.2.2 it follows that $reg(I_1) \leq reg(I)$, thus $(I_1)_{\geq e}$ is stable for $e \geq reg(I)$.

Since $J_0^{sat} = I_1 \cap S_{n_0}$, using iteratively Lemma 1.2.8 it follows that $(J_0^{sat})_{\geq e}$ is stable. Let $e \geq reg(I)$. Since $reg(I) \geq s(J_0^{sat}/J_0) + 1$ it follows that $(J_0)_{\geq e} = (J_0^{sat})_{\geq e}$ is stable. Since $I_0 = J_0 S$, using again Lemma 1.2.8, we get $I_{\geq e}$ stable for $e \geq reg(I)$, as required. □

Theorem 1.2.10 and Proposition 1.2.5 yield the following:

Corollary 1.2.11. *If I is a Borel type ideal, then $reg(I) = min\{e : \ e \geq deg(I)$ and $I_{\geq e}$ is stable$\}$. The same conclusion holds, if I is a monomial ideal with $Ass(S/I)$ totally ordered by inclusion.*

Proof. Denote $f = min\{e : \ I_{\geq e}$ *is stable*$\}$. By 1.2.10, we get $reg(I) \geq f$ and by 1.2.5, $reg(I) \leq f$. For the second statement, by renumbering the variables, we can assume that I is of Borel type. □

Example 1.2.12. Let $I = (x_1^7,\ x_1^5x_2,\ x_1^2x_2^4,\ x_1x_2^6,\ x_1^5x_3^2,\ x_1x_2^4x_3^2) \subset K[x_1,x_2,x_3,x_4]$. We construct the sequential chain of I. We have $I_0 = I$ and $n_0 = m(I_0) = 3$, therefore $J_0 = I_0 \cap K[x_1,x_2,x_3]$. Let $I_1 = (I_0 : x_3^\infty) = (x_1^5, x_1x_2^4)$. We have $n_1 = m(I_1) = 2$, therefore $J_1 = I_1 \cap K[x_1,x_2]$. Let $I_2 = (I_1 : x_2^\infty) = (x_1)$. We have $n_2 = m(I_2) = 1$, therefore $J_2 = I_2 \cap K[x_1]$. One can easily compute, $s(J_0^{sat}/J_0) = 7$, $s(J_1^{sat}/J_1) = 7$ and $s(J_2^{sat}/J_2) = 1$. Using 1.2.2, we get:

$$reg(I) = max\{s(J_0^{sat}/J_0), s(J_1^{sat}/J_1), s(J_2^{sat}/J_2)\} + 1 = 8.$$

We will exemplify the proof of 1.2.10 for I. Let $e \geq 8$ be an integer. Since $(J_2)_{\geq e}$ is obviously stable, it follows from 1.2.8 that $(I_2)_{\geq e} = (J_2S)_{\geq e}$ is also stable. Note that $I_2 = J_1^{sat}S$ and moreover, I_2 and J_1^{sat} have the same minimal set of generators. Therefore, by 1.2.8, it follows that $(J_1^{sat})_{\geq e}$ is stable. Since $e \geq reg(I) > s(J_1^{sat}/J_1) = 7$, it follows that $(J_1)_{\geq e}$ is stable. On the other hand, $I_1 = J_1S$, therefore $(I_1)_{\geq e}$ is stable. Since $J_0^{sat}S = I_1$ we get, from 1.2.8, $(J_0^{sat})_{\geq}$ stable. Since $e \geq reg(I) > s(J_0^{sat}/J_0) = 7$, it follows that $(J_0)_{\geq e}$ is stable. Finally, since $I = I_0 = J_0S$, we obtain $I_{\geq e}$ stable, as required.

Corollary 1.2.13. *If I and J are ideals of Borel type, then*
(a) $reg(I + J) \leq max\{reg(I), reg(J)\}$;
(b) $reg(I \cap J) \leq max\{reg(I), reg(J)\}$.

Proof. Denote $e = max\{reg(I), reg(J)\}$. From Theorem 1.2.10, it follows that $I_{\geq e}$ and $J_{\geq e}$ stable. Therefore, $(I + J)_{\geq e} = I_{\geq e} + J_{\geq e}$ is stable, as a sum of stable ideals. By 1.2.5 it follows that $reg(I + J) \leq e$, thus (a) holds. The proof of (b) is similar. □

Lemma 1.2.14. *Let $I, J \subset S$ be two monomial ideals and let $e \geq deg(I)$ and $f \geq deg(J)$ be two integers such that $I_{\geq e}$ and $J_{\geq f}$ are stable. Then $(I \cdot J)_{\geq e+f}$ is stable.*

Proof. Let $u \in (I \cdot J)_{\geq e+f}$. It follows that $u = v \cdot w$ for some monomials $v \in I$ and $w \in J$. We claim that we can choose v and w such that $v \in I_{\geq e}$ and $w \in J_{\geq e}$.

Indeed, if we write $v = v'a$ for $v' \in G(I)$ and $a \in S$ a monomial and $w = w'b$ for $w' \in G(J)$ and $b \in S$ a monomial, we can find some new monomials $\bar{a}, \bar{b} \in S$ such that $\bar{a}\bar{b} = ab$, $deg(v'\bar{a}) \geq e$ and $deg(w'\bar{b}) \geq f$. We are able to do this, since $e \geq deg(I)$ and $f \geq deg(J)$, so $deg(v') \leq e$ and $deg(w') \leq f$. Changing v with $v'\bar{a}$ and w with $w'\bar{b}$, the claim is proved.

Now, let $j < m(u)$ be an integer and suppose $m(u) = m(v)$. Then $x_j u/x_{m(u)} = (x_j v/x_{m(v)}) \cdot w \in I \cdot J$, because $x_j v/x_{m(v)} \in I$ since $I_{\geq e}$ is stable. Analogously, if $m(u) = m(w)$, then $x_j u/x_{m(u)} = v \cdot (x_j w/x_{m(w)}) \in I \cdot J$. Therefore, $(I \cdot J)_{\geq e+f}$ is stable. $\square$

Theorem 1.2.15. *Let $I, J \subset S$ be two monomial ideals of Borel type. Then*

$$reg(I \cdot J) \leq reg(I) + reg(J).$$

Proof. Since I and J are ideals of Borel type, if we denote $e := reg(I)$ and $f = reg(J)$, by Theorem 1.2.10 it follows that $I_{\geq e}$ and $J_{\geq f}$ are stable. Using the previous lemma, it follows that $(I \cdot J)_{\geq e+f}$ is stable, therefore, using Proposition 1.2.5 we get $reg(I \cdot J) \leq e + f$ as required. $\square$

Corollary 1.2.16. *If $I \subset S$ is an ideal of Borel type, then $reg(I^k) \leq k \cdot reg(I)$.*

Note that there are other large classes of graded ideals which have this property, see for instance [14], but on the other hand, Sturmfels provided an example of a graded ideal $I \subset S$ with $reg(I^2) > 2 \cdot reg(I)$ in [32].

1.3 Monomial ideals of strong Borel type.

Proposition 1.1.2(d) suggested us the following definition.

Definition 1.3.1. *We say that a monomial ideal I is of* strong Borel type *(SBT) if for any monomial $u \in I$ and for any $1 \leq j < i \leq n$, there exists an integer $t \leq \nu_i(u)$ such that $x_j^t u/x_i^{\nu_i(u)} \in I$.*

Remark 1.3.2. Obviously, an ideal of strong Borel type is also an ideal of Borel type, but the converse is not true. Take for instance $I = (x_1^3, x_2^2) \subset K[x_1, x_2]$.

The sum of two ideals of (SBT) is still an ideal of (SBT). Also, the same is true for an intersection or a product of two ideals of (SBT). The proof is similar with the proof of 1.1.3, so we skip it.

Definition 1.3.3. *Let $\mathcal{A} \subset S$ be a set of monomials. We say that I is the* (SBT)-ideal *generated by $\mathcal{A}$, if I is the smallest, with respect to inclusion, ideal of (SBT) containing $\mathcal{A}$. We write $I = SBT(\mathcal{A})$. In particular, if $\mathcal{A} = \{u\}$, where $u \in S$ is a monomial, we say that I is the* principal (SBT)-ideal *generated by u, and we write $I = SBT(u)$.*

Lemma 1.3.4. *Let $1 \leq i_1 < i_2 < \cdots < i_r \leq n$ be some integers, $\alpha_1, \ldots, \alpha_r$ some positive integers and $u = x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_r}^{\alpha_r} \in S$. Then, the principal (SBT)-ideal generated by u, is:*

$$I = SBT(u) = \prod_{q=1}^{r} (\mathbf{m}_q^{[\alpha_q]}), \text{ where } \mathbf{m}_q = \{x_1, \ldots, x_{i_q}\} \text{ and } \mathbf{m}_q^{[\alpha_q]} = \{x_1^{\alpha_q}, \ldots, x_{i_q}^{\alpha_q}\}.$$

Proof. Denote $I' = \prod_{q=1}^{r} (\mathbf{m}_q^{[\alpha_q]})$. If $v \in G(I')$, then $v = x_{j_1}^{\alpha_1} x_{j_2}^{\alpha_2} \cdots x_{j_r}^{\alpha_r}$, for some $1 \leq j_q \leq i_q$, where $1 \leq q \leq r$. Since

$$v = \frac{x_{j_r}^{\alpha_r}}{x_{i_r}^{\alpha_r}} \cdots \frac{x_{j_2}^{\alpha_2}}{x_{i_2}^{\alpha_2}} \cdot \frac{x_{j_1}^{\alpha_1}}{x_{i_1}^{\alpha_1}} u,$$

and I is of (SBT) it follows that $v \in I$ and thus $I' \subseteq I$. For the converse, simply notice that I' is itself a (SBT)-ideal. Therefore $I = SBT(u)$ as required. □

Remark 1.3.5. *Let $1 \leq i_1 < i_2 < \cdots < i_r \leq n$ be some integers and $\alpha_1, \ldots, \alpha_r$ some positive integers and $u = x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_r}^{\alpha_r} \in S$. Let $I = SBT(u)$. We describe the sequential chain of I. Denote $I_r := I$. Since $I_r = I = \prod_{q=1}^{r} (\mathbf{m}_q^{[\alpha_q]})$, it follows that $I_{r-1} := (I_r : x_{i_r}^{\infty}) = \prod_{q=1}^{r-1} (\mathbf{m}_q^{[\alpha_q]})$. Analogously, we get $I_q = (I_{q+1} : x_{i_{q+1}}^{\infty}) = \prod_{e=1}^{q} (\mathbf{m}_e^{[\alpha_e]})$, for all $0 \leq q < r$. Therefore, the sequential chain of I is,*

$$I = I_r \subset I_{r-1} \subset \cdots \subset I_1 \subset I_0 = S.$$

Let J_q be the ideal in $S_q = K[x_1, \ldots, x_{i_q}]$ generated by $G(I_q)$, for $1 \leq q \leq r$. If $s_q = s(J_q^{sat}/J_q)$, 1.2.2 implies $reg(I) = max\{s_q : 1 \leq q \leq r\}$

Our next goal is to compute the regularity of a principal (SBT)-ideal, in a special case. In order to do so, we will compute the sequential chain of I and than apply Proposition 1.2.2.

Theorem 1.3.6. *Let $1 \leq i_1 < i_2 < \cdots < i_r \leq n$ be some integers, $\alpha_1 \geq \alpha_1 \geq \cdots \geq \alpha_r$ some positive integers and $u = x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_r}^{\alpha_r} \in S$. Let $I = SBT(u)$. For $1 \leq q \leq r$, we define the numbers:*

$$\chi_q = \alpha_1 + \cdots + \alpha_{q-1} + (\alpha_q - 1) i_q.$$

With the above notations, we have $reg(SBT(u)) = \max_{q=1}^{r} \chi_q + 1$.

Proof. We use the notations from 1.3.5. In order to compute the regularity of I we must determine the numbers $s_q := s(J_q^{sat}/J_q)$. We will prove that $s_q = \chi_q$. First of all, note that $J_q = I_q \cap S_q$ and $J_q^{sat} = I_{q-1} \cap S_q$. We fix $1 \leq q \leq r$ and we denote $A := \{1, 2, \ldots, i_q\} \setminus \{i_1, \ldots, i_{q-1}\}$. Let

$$w = \prod_{e=1}^{q-1} x_{i_e}^{\alpha_e + \alpha_q - 1} \cdot \prod_{j \in A} x_j^{\alpha_q - 1} \in S_q.$$

Obviously, $deg(w) = \chi_q$. We claim that $w \in J_q^{sat}$, but $w \notin J_q$ and, therefore, $s_q \geq \chi_q$. Indeed, since $\prod_{e=1}^{q-1} x_{i_e}^{\alpha_e + \alpha_q - 1} \in J_q^{sat} = \prod_{e=1}^{q-1} (\mathbf{m}_e^{[\alpha_e]}) \subset S_q$ it follows that $w \in J_q^{sat}$.

We assume, by contradiction, that $w \in J_q$. Thus, $w = x_{j_1}^{\alpha_1} \cdots x_{j_q}^{\alpha_q} y$, where $1 \leq j_e \leq i_e$ for all $e \in \{1, \dots, q\}$ and $y \in S$ is a monomial. We claim that $\{j_1, j_2, \cdots, j_{q-1}\} = \{i_1, \dots, i_{q-1}\}$. Let $k \in \{1, \dots, q-1\}$. If $j_k \in A$, since $x_{j_k}^{\alpha_k} | w$ it follows that $\alpha_k \leq \alpha_q - 1$, a contradiction, since $\alpha_k \geq \alpha_q$. Therefore, $j_k \in \{i_1, \dots, i_{q-1}\}$.

Using induction on $1 \leq e \leq q-1$, we prove that $\{j_1, \dots, j_e\} = \{i_1, \dots, i_e\}$ for all e. Let $e = 1$. Since $x_{j_1}^{\alpha_1} | w$, $j_1 \leq i_1$ and for all $j < i_1$ we have $j \in A$, it follows that $j_1 = i_1$. Suppose $e \leq q-1$ and $\{j_1, \dots, j_{e-1}\} = \{i_1, \dots, i_{e-1}\}$. We have $j_e \leq i_e$. Suppose $j_e < i_e$. Since $x_{j_e}^{\alpha_e} | w$ it follows that $j_e = i_k$ for some $k < e$. So $w = (x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k} \cdots x_{i_{e-1}}^{\alpha_{e-1}})(x_{j_e}^{\alpha_e} \cdots x_{j_q}^{\alpha_q} \cdot y)$ and it follows $x_{i_k}^{\alpha_e + \alpha_k} | w$. But this is false, since $\alpha_e + \alpha_k > \alpha_k + \alpha_q - 1 = deg_{x_{i_k}}(w) =$ the exponent in x_{i_k} of w. Thus, $j_e = i_e$ and the induction holds.

We get $w = \prod_{e=1}^{q-1} x_{i_e}^{\alpha_e} \cdot x_{j_q}^{\alpha_q} \cdot y$. If $j_q \in A$, obviously, we get a contradiction. Thus $j_q \in \{i_1, \dots, i_{q-1}\}$. But this, again, cannot be true, since if $j_q = i_e$, it follows that $x_{i_e}^{\alpha_q + \alpha_e} | w$. In conclusion, our assumption is false and thus $w \notin J_q$.

In order to prove that $s_q \leq \chi_q$, we choose a monomial $w \in J_q^{sat}$ such that $w \notin J_q$ and we show that $deg(w) \leq \chi_q$. Since $w \in J_q^{sat}$, it follows that

$$w = \prod_{e=1}^{q-1} x_{j_e}^{\alpha_e} \prod_{j=1}^{i_q} x_j^{\beta_j},$$

where $1 \leq j_e \leq i_e$ and β_j are some nonnegative integers. Since $w \notin J_q$ it follows, obviously, that $\beta_j \leq \alpha_q - 1$ and, therefore, $deg(w) = \alpha_1 + \cdots + \alpha_{q-1} + \sum_{j=1}^{i_q} \beta_j \leq \chi_q$, as required. □

Example 1.3.7. Let $u = x_2^7 x_3^6 \in S = K[x_1, x_2, x_3]$. We have $i_1 = 2$, $i_2 = 3$, $\alpha_1 = 7$ and $\alpha_2 = 6$. From Lemma 1.3.4 it follows that $I = SBT(u) = (x_1^7, x_2^7)(x_1^6, x_2^6, x_3^6)$. We have $J_1 = (x_1^7, x_2^7) \subset K[x_1, x_2]$ and $J_2 = I$. Also, $J_1^{sat} = K[x_1, x_2]$ and $J_2^{sat} = (x_1^7, x_2^7) \subset S$. $\chi_1 = (\alpha_1 - 1) \cdot 2 = 12$. $\chi_2 = \alpha_1 + (\alpha_2 - 1) \cdot 3 = 7 + 15 = 22$. By 1.3.6, $reg(I) = max\{12, 22\} + 1 = 23$.

1.4 d-fixed ideals.

In the following $\mathbf{d} : 1 = d_0 | d_1 | \cdots | d_s$ is a strictly increasing sequence of positive integers, where s is a positive integer. We say that $\mathbf{d}$ is a $\mathbf{d}$-sequence. We can take also $s = +\infty$, but for convenience we will not do it.

Lemma 1.4.1. *Let $\mathbf{d}$ be a $\mathbf{d}$-sequence. Then, for any $a \in \mathbb{N}$, there exists an unique sequence of positive integers $a_0, a_1, \dots, a_s$ such that:*

1. $a = \sum_{t=0}^{s} a_t d_t$ *and*
2. $0 \leq a_t < \frac{d_{t+1}}{d_t}$, *for any* $0 \leq t < s$.

Conversely, if $\mathbf{d} : 1 = d_0 < d_1 < \cdots < d_s$ is a sequence of positive integers such that for any $a \in \mathbb{N}$ there exists an unique sequence of positive integers $a_0, a_1, \dots, a_s$ as before, then $\mathbf{d}$ is a $\mathbf{d}$-sequence.

Proof. Let a_s be the quotient of a divided by d_s. For $0 \le t < s$ let a_t be the quotient of $(a - q_{t+1})$ divided by d_t, where $q_{t+1} = \sum_{j=t+1}^{s} a_j d_j$. We will prove that $a_0, a_1, \ldots, a_s$ fulfill the required conditions. Indeed, it is obvious that $a = \sum_{t=0}^{s} a_t d_t$. On the other hand, $a - q_{t+1} < d_{t+1}$, therefore, since a_t is the quotient of $(a - q_{t+1})$ divided by d_t, it follows that $a_t < \frac{d_{t+1}}{d_t}$.

Suppose there exists another decomposition $a = \sum_{j=0}^{s} b_j d_j$ which also fulfill the conditions 1 and 2. Then, we may assume that there exists an integer $0 \le t \le s$ such that $b_s = a_s, \cdots, b_{t+1} = a_{t+1}$ and $b_t > a_t$. Notice that $d_t > \sum_{j=0}^{t-1} a_j d_j$. Indeed,

$$\sum_{j=0}^{t-1} a_j d_j \le \sum_{j=0}^{t-1} (\frac{d_{j+1}}{d_j} - 1) d_j = (d_1 - d_0) + (d_2 - d_1) + \cdots + (d_t - d_{t-1}) = d_t - 1 < d_t.$$

We have $0 = \sum_{j=0}^{s} (b_j - a_j) d_j = \sum_{j=0}^{t} (b_j - a_j) d_j$, but on the other hand:

$$(b_t - a_t) d_t \ge d_t > \sum_{j=0}^{t-1} a_j d_j \ge \sum_{j=0}^{t-1} (a_j - b_j) d_j$$

and therefore $(b_t - a_t) d_t - \sum_{j=0}^{t-1} (a_j - b_j) d_j = \sum_{j=0}^{t} (b_j - a_j) d_j > 0$, which is a contradiction.

For the converse, we use induction on $0 \le t < s$, the assertion being obvious for $t = 0$. Suppose $t > 0$ and $d_0 | d_1 | \cdots | d_t$ and consider the decomposition of $d_{t+1} - 1$. Since $d_{t+1} - 1 < d_{t+1}$, it follows that $d_{t+1} - 1 = \sum_{j=0}^{t} a_t d_t$. On the other hand, since $d_{t+1} - 1$ is the largest integer less than d_{t+1}, each a_j is maximal between the integers $< d_{j+1}/d_j$, for $j < t$. Therefore $a_j = d_{j+1}/d_j - 1$ for $0 \le j < t$. Thus:

$$d_{t+1} = 1 + d_{t+1} - 1 = 1 + a_0 d_0 + a_1 d_1 + \cdots + a_t d_t = d_1 + a_1 d_1 + a_2 d_2 + \cdots + a_t d_t =$$

$$= d_2 + a_2 d_2 + \cdots + a_t d_t = \cdots = (a_t + 1) d_t, \quad so \ \ d_t | d_{t+1}.$$

□

Definition 1.4.2. *Let a, b be two positive integers and consider the* $\mathbf{d}$*-decompositions $a = \sum_{j=0}^{s} a_j d_j$ and $b = \sum_{j=0}^{s} b_j d_j$. We say that $a \le_{\mathbf{d}} b$ if $a_j \le b_j$ for any $0 \le j \le s$.*

Definition 1.4.3. *We say that a monomial ideal $I \subset S = k[x_1, \ldots, x_n]$ is* $\mathbf{d}$*-fixed, if for any monomial $u \in I$ and for any indices $1 \le j < i \le n$, if $t \le_{\mathbf{d}} \nu_i(u)$ (where $\nu_i(u)$ denotes the exponent of the variable x_i in u) then $u \cdot x_j^t / x_i^t \in I$.*

Example 1.4.4. *(a) Let $\mathbf{d} : 1 | p | p^2 | p^3 | \cdots$, where $p > 0$ is a prime number. Then a* $\mathbf{d}$*-fixed ideal I is a p-Borel ideal. Therefore, definition* 1.4.3 *generalize the definition of a p-Borel ideal.*

(b) Suppose $\mathbf{d} : 1$, i.e. $s = 0$. Let $I \subset S$ be a monomial ideal. Then I is $\mathbf{d}$*-fixed, if and only if I is strongly stable. Indeed, in the definition of a* $\mathbf{d}$*-fixed ideal, we can always choose $t = 1$, because the* $\mathbf{d}$*-decomposition of any positive integer m is in this case $m = m \cdot 1$.*

(c) Let $\mathbf{d} : 1 = d_0 | d_1 | \cdots | d_s$ be a $\mathbf{d}$*-sequence. If $I, J \subset S$ are* $\mathbf{d}$*-fixed ideals, then $I + J$ and $I \cap J$ are also* $\mathbf{d}$*-fixed ideals. This is obvious from the definition.*

Lemma 1.4.5. *Let a, b be two positive integers with $a \leq_{\mathbf{d}} b$. Suppose $b = b' + b''$, where b' and b'' are positive integers. Then, there exists some positive integers $a' \leq_{\mathbf{d}} b'$ and $a'' \leq_{\mathbf{d}} b''$ such that $a = a' + a''$.*

Proof. Let $a = \sum_{t=0}^{s} a_t d_t$, $b = \sum_{t=0}^{s} b_t d_t$, $b' = \sum_{t=0}^{s} b'_t d_t$, $b'' = \sum_{t=0}^{s} b''_t d_t$. The hypothesis implies $a_t \leq b_t < d_{t+1}/d_t$ and $b'_t, b''_t < d_{t+1}/d_t$ for any $0 \leq t < s$. We construct the sequences a'_t, a''_t using decreasing induction on t. Suppose we have already defined a'_j, a''_j for $j > t$ such that $\sum_{i=j}^{s}(a'_i + a''_i)d_i = \sum_{i=j}^{s} a_i d_i$ and $b_{t+1} = b'_{t+1} + b''_{t+1}$. This is obvious for $t = s$.

We consider two cases. If $b_t = b'_t + b''_t$, then we choose $a'_t \leq b'_t$ and $a''_t \leq b''_t$ such that $a'_t + a''_t = a_t$. We can do this, because $a_t \leq b_t$. Also, it is obvious from the induction hypothesis that $\sum_{i=t}^{s}(a'_i + a''_i)d_i = \sum_{i=t}^{s} a_i d_i$, so we can pass from t to $t-1$.

If $b_t \neq b'_t + b''_t$ we claim that $b'_t + b''_t = b_t - 1$. Indeed, $\sum_{j=0}^{t-1}(b'_j + b''_j)d_j < 2d_t$ and therefore it is impossible to have $b'_t + b''_t \leq b_t - 2$, otherwise $\sum_{j=0}^{t}(b'_j + b''_j)d_j < b_t d_t$ and we contradict the equality $b = b' + b''$. Also, since $b'_{t+1} + b''_{t+1} = b_{t+1}$, we cannot have $b'_t + b''_t > b_t$. Similarly we get $b'_{t-1} + b''_{t-1} > b_{t-1}$. By recurrence, we conclude that there exists an integer $u < t$ such that: $b'_{u-1} + b''_{u-1} = b_{u-1}$, $,b'_u + b''_u = b_u + d_{u+1}/d_u$, $b'_{u+1} + b''_{u+1} = b_{u+1} + d_{u+2}/d_{u+1} - 1$, $\ldots$, $b'_{t-1} + b''_{t-1} = b_{t-1} + d_t/d_{t-1} - 1$.

If $a_j = b_j$ for any $j \in \{u, \ldots, t\}$, we simply choose $a'_j = b'_j$ and $a''_j = b''_j$ for any $j \in \{u, \ldots, t\}$ and the required conditions are fulfilled, so we can pass from t to $u-1$. If this is not the case, then there exists an integer $u \leq q \leq t$ such that $a_t = b_t, \ldots, a_{q+1} = b_{q+1}$ and $a_q < b_q$. If $q = t$ then for any $j \in \{u, \ldots, t\}$ we can choose $a'_j \leq b'_j$ and $a''_j \leq b''_j$ such that $a'_j + a''_j = a_j$. For $j < t$ the previous assertion is obvious because $b'_j + b''_j \geq b_j$, and for $j = t$, since $a_t < b_t$ we have in fact $a_t \leq b'_t + b''_t = b_t - 1$ and therefore we can choose again a'_t and a''_t. The conditions are satisfied so we can pass from t to $u-1$.

Suppose $q < t$. For $j \in \{u, \ldots, q-1\}$ we choose $a'_j \leq b'_j$ and $a''_j \leq b''_j$ such that $a'_j + a''_j = a_j$. We can do this because $b'_j + b''_j \geq b_j \geq a_j$. We choose a'_q and a''_q such that $a'_q + a''_q = a_q + d_{q+1}/d_q$. We can make this choice, because $a_q \leq b_q - 1$ and $b'_q + b''_q \geq b_q + d_{q+1}/d_q - 1$. For $j > q$, we simply put $a'_j = b'_j$ and $a''_j = b''_j$. To pass from t to $u-1$ is enough to see that $\sum_{j=u}^{t} a_j d_j = \sum_{j=u}^{t}(a'_j + a''_j)d_j$. Indeed,

$$\sum_{j=u}^{t}(a'_j + a''_j)d_j = \sum_{j=u}^{q-1}(a'_j + a''_j)d_j + (a'_q + a''_q)d_q + \sum_{j=q+1}^{t}(a'_j + a''_j)d_j =$$

$$= \sum_{j=u}^{q-1} a_j d_j + (a_q + d_{q+1}/d_q)d_q + \sum_{j=q+1}^{t-1}(a_j + d_{j+1}/d_j - 1)d_j + (a_t - 1)d_t =$$

$$= \sum_{j=u}^{t} a_j d_j + d_{q+1} + \sum_{j=q+1}^{t-1}(d_{j+1} - d_j) - d_t = \sum_{j=u}^{t} a_j d_j,$$

The induction ends when $t = -1$. Finally, we obtain a' and a'' such that $a' + a'' = a$, $a'_t \leq b'_t$ and $a''_t \leq b''_t$, as required. $\square$

Corollary 1.4.6. *If $I, J \subset S$ are* **d**-*fixed ideals then $I \cdot J$ is a* **d**-*fixed ideal.*

Proof. Let $u \in I \cdot J$ be a monomial and let $1 \leq j < i \leq n$ be two integers. We can write $u = v \cdot w$, where $v \in I$ and $w \in J$ are monomials. Let $t \leq_{\mathbf{d}} \nu_i(u)$ be a positive integer. Since $\nu_i(u) = \nu_i(v) + \nu_i(w)$, by previous lemma, we can choose two positive integers $t' \leq_{\mathbf{d}} \nu_i(v)$ and $t'' \leq_{\mathbf{d}} \nu_i(w)$ such that $t = t' + t''$. Since I and J are **d**-fixed, it follows that $x_j^{t'} v / x_i^{\nu_i(v)} \in I$ and $x_j^{t''} w / x_i^{\nu_i(w)} \in J$. Therefore, $x_j^t u / x_i^{\nu_i(u)} \in I \cdot J$ and thus $I \cdot J$ is **d**-fixed, as required. □

Definition 1.4.7. *A* **d**-*fixed ideal I is called* principal *if it is generated, as a* **d**-*fixed ideal by one monomial $u \in S$, i.e. I is the smallest* **d**-*fixed ideal which contains u. We write $I = < u >_{\mathbf{d}}$. More generally, if $u_1, \ldots, u_r \in S$ are monomials, the* **d**-*fixed ideal generated by $u_1, \ldots, u_r$ is the smallest* **d**-*fixed ideal I which contains $u_1, \ldots, u_r$. We write $I = < u_1, \ldots, u_r >_{\mathbf{d}}$.*

Our next goal is to describe the principal **d**-fixed ideals. The easiest case is when we have a **d**-fixed ideal generated by the power of a variable. We denote $\mathbf{m} = \{x_1, \ldots, x_n\}$ and $\mathbf{m}^{[d]} = \{x_1^d, \ldots, x_n^d\}$, where d is a positive integer. We have the following proposition.

Proposition 1.4.8. *If $u = x_n^{\alpha}$, then $I = < u >_{\mathbf{d}} = \prod_{t=0}^{s} (\mathbf{m}^{[d_t]})^{\alpha_t}$, where $\alpha = \sum_{t=0}^{s} \alpha_t d_t$.*

Proof. Let $I' = \prod_{t=0}^{s} (\mathbf{m}^{[d_t]})^{\alpha_t}$. The minimal generators of I' are monomials of the type $w = \prod_{t=0}^{s} \prod_{j=1}^{n} x_j^{\lambda_{tj} \cdot d_t}$, where $0 \leq \lambda_{tj}$ and $\sum_{j=1}^{n} \lambda_{tj} = \alpha_t$. First, let us show that $I' \subset I$. In order to do this, we choose w a minimal generator of I' (the one bellow). We write x_n^{α} like this: $x_n^{\alpha} = x_n^{\alpha_0 d_0 + \alpha_1 d_1 + \cdots + \alpha_s d_s} = x_n^{\alpha_0 d_0} \cdot x_n^{\alpha_1 d_1} \cdots x_n^{\alpha_s d_s}$. Since $\lambda_{01} d_0 \leq_{\mathbf{d}} \alpha_0 d_0 + \alpha_1 d_1 + \cdots + \alpha_s d_s$ and I is **d**-fixed it follows that $x_1^{\lambda_{01} d_0} x_n^{\alpha - \lambda_{01} d_0} \in I$. Also, $\lambda_{02} d_0 < \alpha - \lambda_{01} d_0 = (\alpha_0 - \lambda_{01}) d_0 + \alpha_1 d_1 + \cdots + \alpha_s d_s$, and since I is **d**-fixed it follows that $x_1^{\lambda_{01} d_0} x_2^{\lambda_{02} d_0} x_n^{\alpha - \lambda_{01} d_0 - \lambda_{02} d_0} \in I$. Using iteratively this argument, one can easily see that $x_1^{\lambda_{01} d_0} \cdots x_n^{\lambda_{0n} d_0} x_n^{\alpha - \alpha_0 d_0} \in I$. Also $\alpha - \alpha_0 d_0 = \alpha_1 d_1 + \cdots + \alpha_n d_n$. Again, using an inductive argument, we get:

$$(x_1^{\lambda_{01} d_0} \cdots x_n^{\lambda_{0n} d_0}) \cdot (x_1^{\lambda_{11} d_1} \cdots x_n^{\lambda_{1n} d_1}) \cdots (x_1^{\lambda_{s1} d_s} \cdots x_n^{\lambda_{sn} d_s}) = w \in I.$$

For the converse, i.e. $I \subset I'$, is enough to verify that I' is **d**-fixed. In order to do this, is enough to prove that the minimal generators of I' fulfill the definition of a **d**-fixed ideal. Let w be a minimal generator of I'. Let $2 \leq i \leq n$. Then $\nu_i(w) = \sum_{t=0}^{s} \lambda_{ti} d_t$. If $\beta \leq_{\mathbf{d}} \nu_i(w)$ then $\beta = \sum_{t=0}^{s} \beta_t d_t$ with $\beta_t \leq \lambda_{ti}$. Let $1 \leq k < i$. We have

$$w \cdot x_k^{\beta} / x_i^{\beta} = \prod_{t=0}^{s} (\prod_{j \neq i,k} x_j^{\lambda_{tj} d_t}) \cdot x_i^{(\lambda_{ti} - \beta_t) d_t} \cdot x_k^{(\lambda_{tk} + \beta_t) d_t}.$$

Thus $w \cdot x_k^{\beta} / x_i^{\beta} \in I'$ and therefore I' is **d**-fixed. Since I is the smallest **d**-fixed ideal which contains x_n^{α} it follows that $I \subset I'$. □

Proposition 1.4.9. *If $\alpha \leq \beta$ then $< x_n^{\beta} >_{\mathbf{d}} \subseteq < x_n^{\alpha} >_{\mathbf{d}}$.*

Proof. The case $\alpha = \beta$ is obvious, so we may assume $\alpha < \beta$. We denote $I =< x_n^{\alpha} >_{\mathbf{d}}$ and $I' =< x_n^{\beta} >_{\mathbf{d}}$. We write $\alpha = \sum_{t=0}^{s} \alpha_t d_t$ and $\beta = \sum_{t=0}^{s} \beta_t d_t$. If w is a minimal generator of I' then $w = \prod_{t=0}^{s} \prod_{i=1}^{n} x_i^{\lambda_{ti} d_t}$, where $0 \leq \lambda_{ti}$ and $\sum_{i=1}^{n} \lambda_{ti} = \beta_t$. We claim that $w \in I$ and therefore $I' \subset I$ as required.

Since $\alpha < \beta$ there exists $t \in \{0, \ldots, s\}$ such that $\alpha_s = \beta_s, \ldots, \alpha_{t+1} = \beta_{t+1}$ and $\alpha_t < \beta_t$. We may assume some $\lambda_{tk} > 0$. We have

$$w = \prod_{j=0}^{s} \prod_{i=1}^{n} x_i^{\lambda_{ji} d_j} = \prod_{j=0}^{t-1} x_k^{\alpha_j d_j} x_k^{(\lambda_{tk}-1) d_t} x_k^{d_t - \sum_{j=0}^{t-1} \alpha_j d_j} \prod_{i \neq k}^{n} x_i^{\lambda_{ti} d_t} \prod_{j>t} \prod_{i=1}^{n} x_i^{\lambda_{ji} d_j}$$

and now it is obvious that $w \in I$. □

Proposition 1.4.10. *If α and β are two positive integers, then $< x_n^{\alpha+\beta} >_{\mathbf{d}} \subseteq < x_n^{\alpha} >_{\mathbf{d}} \cdot < x_n^{\beta} >_{\mathbf{d}}$. The equality holds if and only if $\alpha_t + \beta_t < d_{t+1}/d_t$ for all $0 \leq t \leq s$, where $\alpha = \sum_{t=0}^{s} \alpha_t d_t$ and $\beta = \sum_{t=0}^{s} \beta_t d_t$*

Proof. We denote $I =< x_n^{\alpha+\beta} >$ and $I' =< x_n^{\alpha} > \cdot < x_n^{\beta} >$. Also, we denote $\gamma := \alpha + \beta$. Let $\gamma = \sum_{t=0}^{s} \gamma_t d_t$. We may assume $\gamma_s \neq 0$, otherwise, we replace s with $s' = max\{t |\ \gamma_t \neq 0\}$. We use induction on $t = min\{j |\ \alpha_j^2 + \beta_j^2 \neq 0\}$. It $t = s$, it follows that $\alpha = \alpha_s d_s$, $\beta = \beta_s d_s$ and thus $I = I'$. Suppose $t < s$. We should consider two cases: (I) $\alpha_t + \beta_t = \gamma_t$ or (II) $\alpha_t + \beta_t = \gamma_t + d_{t+1}/d_t$.

(I) Let $\bar{\alpha} := \alpha - \alpha_t d_t$ and $\bar{\beta} = \beta - \beta_t d_t$. Let $\bar{\gamma} := \bar{\alpha} + \bar{\beta}$. We denote $\bar{I} :=< x_n^{\bar{\gamma}} >_{\mathbf{d}}$ and $\bar{I}' :=< x_n^{\bar{\alpha}} >_{\mathbf{d}} \cdot < x_n^{\bar{\beta}} >_{\mathbf{d}}$. By induction hypothesis, we have $\bar{I} \subseteq \bar{I}'$. Therefore, since $I = (\mathbf{m}^{[d_t]})^{\gamma_t} \bar{I}$ and $I' = (\mathbf{m}^{[d_t]})^{\gamma_t} \bar{I}'$ it follows that $I \subseteq I'$ as required.

(II)As above, we define $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, $\bar{I}$ and $\bar{I}'$. We notice that $I = (\mathbf{m}^{[d_t]})^{\alpha_t + \beta_t - d_{t+1}/d_t} \cdot (\mathbf{m}^{[d_{t+1}]}) \bar{I}$ and $I' = (\mathbf{m}^{[d_t]})^{\alpha_t + \beta_t} \bar{I}'$. Using induction hypothesis, it follows that $I \subset I'$. Note that in this case, the inclusion is strict, therefore we get the second statement of the proposition. □

We have the general description of a principal **d**-fixed ideal given by the following proposition. In the proof, we will apply Lemma 1.4.5.

Proposition 1.4.11. *Let $1 \leq i_1 < i_2 < \cdots < i_r = n$ and let $\alpha_1, \ldots, \alpha_r$ be some positive integers. If $u = x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_r}^{\alpha_r}$ then:*

$$I =< u >_{\mathbf{d}} =< x_{i_1}^{\alpha_1} >_{\mathbf{d}} \cdot < x_{i_2}^{\alpha_2} >_{\mathbf{d}} \cdots < x_{i_r}^{\alpha_r} >_{\mathbf{d}} = \prod_{q=1}^{r} \prod_{t=0}^{s} (\mathbf{m}_q^{[d_t]})^{\alpha_{qt}},$$

where $\mathbf{m}_q = \{x_1, \ldots, x_{i_q}\}$, $\mathbf{m}_q^{[d_t]} = \{x_1^{d_t}, \ldots, x_{i_q}^{d_t}\}$ and $\alpha_q = \sum_{t=0}^{s} \alpha_{qt} d_t$.

Proof. Let $I' = \prod_{q=1}^{r} \prod_{t=0}^{s} (\mathbf{m}_q^{[d_t]})^{\alpha_{qt}}$. The minimal generators of I' are monomials of the type $w = \prod_{q=1}^{r} \prod_{t=0}^{s} \prod_{j=1}^{i_q} x_j^{\lambda_{qtj} \cdot d_t}$, where $0 \leq \lambda_{qtj}$ and $\sum_{j=1}^{n} \lambda_{qtj} = \alpha_{qt}$. First, we show that

$I' \subset I$. In order to do this, it is enough to prove that by iterative transformations we can modify u such that we obtain w.

The idea of this transformations is the same as in the proof of 1.4.8. Without given all the details, one can see that if we rewrite u as

$$(x_{i_1}^{\alpha_{10}d_0}x_{i_1}^{\alpha_{11}d_1}\cdots x_{i_1}^{\alpha_{1s}d_s})\cdots(x_{i_r}^{\alpha_{r0}d_0}x_{i_r}^{\alpha_{r1}d_1}\cdots x_{i_r}^{\alpha_{rs}d_s}),$$

where $\alpha_q = \sum_{t=0}^{s}\alpha_{qt}d_t$, we can pass to w, using the transformations

$$x_{i_1}^{\alpha_{10}d_0} \mapsto \prod_{j=1}^{i_1} x_j^{\lambda_{10j}d_0}, \ldots, x_{i_1}^{\alpha_{1s}d_s} \mapsto \prod_{j=1}^{i_1} x_j^{\lambda_{1sj}d_s}, \ldots, x_{i_r}^{\alpha_{r0}d_0} \mapsto \prod_{j=1}^{i_r} x_j^{\lambda_{r0j}d_0}, \ldots, x_{i_r}^{\alpha_{rs}d_s} \mapsto \prod_{j=1}^{i_r} x_j^{\lambda_{rsj}d_s}.$$

Therefore $w \in I$, and thus $I' \subset I$. For the converse, it is enough to see that I' is a **d**-fixed ideal. Let w be a minimal generator of I'. We choose an index $2 \leq i \leq n$. Then $\nu_i(w) = \sum_{q=1}^{r}\sum_{t=0}^{s}\lambda_{qti}d_t$. Let $\beta \leq \nu_i(w)$. Using Lemma 1.4.5, we can choose some positive integers $\beta_1, \ldots, \beta_r$ such that:

$$(a)\beta = \sum_{q=1, i_q \geq i}^{r} \beta_q \ \mathit{and}\ (b)\beta_q \leq_{\mathbf{d}} \sum_{t=0}^{s}\lambda_{qti}d_t,$$

i.e. $\beta_{qt} \leq \lambda_{qti}$, where $\beta_q = \sum_{t=0}^{s}\beta_{qt}d_t$. Let $k < i$. Then,

$$w \cdot x_k^{\beta}/x_i^{\beta} = \prod_{q=1}^{r}\prod_{t=0}^{s}\left(\prod_{j=1, j\neq k,i}^{i_q} x_j^{\lambda_{qtj}\cdot d_t}\right) x_i^{(\lambda_{qti}-\beta_{qt})d_t} x_k^{(\lambda_{qtk}+\beta_{qt})d_t}.$$

Now, it is easy to see that $w \cdot x_k^{\beta}/x_i^{\beta} \in I'$, and therefore I' is **d**-fixed. □

Example 1.4.12. Let $\mathbf{d} : 1|2|4|12$.

1. Let $u = x_3^{21}$. We have $21 = 1\cdot 1 + 0\cdot 2 + 2\cdot 4 + 1\cdot 12$. From 1.4.8, we get:
$$<u>_{\mathbf{d}} = (x_1, x_2, x_3)(x_1^4, x_2^4, x_3^4)^2(x_1^{12}, x_2^{12}, x_3^{12}).$$
2. Let $u = x_1^2x_2^9x_3^{16}$. We have $9 = 1\cdot 1 + 2\cdot 4$ and $16 = 1\cdot 4 + 1\cdot 12$. From 1.4.11, we get
$$<u>_{\mathbf{d}} = x_1^2 <x_2^9>_{\mathbf{d}}<x_3^{16}>_{\mathbf{d}} = x_1^2(x_1, x_2)(x_1^4, x_2^4)^2(x_1^4, x_2^4, x_3^4)(x_1^{12}, x_2^{12}, x_3^{12}).$$

Remark 1.4.13. Any **d**-fixed ideal I is a Borel type ideal. Indeed, Proposition 1.1.4(d) says that an ideal I is of Borel type if and only if for any $1 \leq j < i \leq n$, there exists an positive integer t such that $x_j^t(u/x_i^{\nu_i(u)}) \in I$. Choosing $t = \nu_i(u)$, is easy to see that the definition of a **d**-fixed ideal implies the condition above.

Let $u = x_{i_1}^{\alpha_1}x_{i_2}^{\alpha_2}\cdots x_{i_r}^{\alpha_r}$ and $I = <u>_{\mathbf{d}} = \prod_{q=1}^{r}\prod_{t=0}^{s}(\mathbf{m}_q^{[d_t]})^{\alpha_{qt}}$, where $\alpha_q = \sum_{t=0}^{s}\alpha_{qt}d_t$. Let $I_{r-e} = \prod_{q=1}^{e}\prod_{t=0}^{s}(\mathbf{m}_q^{[d_t]})^{\alpha_{qt}}$. Then $I = I_0 \subset I_1 \subset \cdots \subset I_r = S$ is the sequential chain of I. Let $n_\ell = i_{q_{r-\ell}}$. Indeed, since $x_{n_\ell}^{\alpha_{r-e}}I_{\ell+1} \subset I_\ell \Rightarrow I_{\ell+1} \subset (I_\ell : x_{n_\ell}^{\infty})$. For the converse, let $w \in (I_\ell : x_{n_\ell}^{\infty})$ be any minimal generator. Then there exists an integer b such that $w \cdot x_{n_\ell}^b \in I_\ell$. We may assume that w is a minimal generator of I_ℓ. Then $w \cdot x_{n_\ell}^b = w' \cdot y$ for a $w' \in I_{\ell+1}$ and $y \in \prod_{j=0}^{t}(\mathbf{m}_{r-\ell}^{[d_j]})^{\alpha_{r-\ell,j}}$ with $x_{n_\ell}^b | y$. Thus $w'|w$, and therefore $w \in I_{\ell+1}$.

1.5 Socle of factors by principal d-fixed ideals.

In the following, we suppose $n \geq 2$.

Lemma 1.5.1. *Let* $\mathbf{d} : 1 = d_0|d_1|\cdots|d_s$, $\alpha \in \mathbb{N}$ *and* $I =< x_n^\alpha >_{\mathbf{d}} = \prod_{t=0}^s (\mathbf{m}^{[d_t]})^{\alpha_t}$. *Let* $q_t = \sum_{j=t}^s \alpha_j d_j$. *Let*

$$J = \sum_{t=0,\alpha_t>0}^{s} (x_1\cdots x_n)^{d_t-1}(\mathbf{m}^{[d_t]})^{\alpha_t-1}\prod_{j>t}(\mathbf{m}^{[d_j]})^{\alpha_j}.$$

Then:

1. $Soc(S/I) = \frac{J+I}{I}$
2. *Let* e *be a positive integer. Then* $(\frac{J+I}{I})_e \neq 0 \Leftrightarrow e = q_t + (n-1)(d_t-1)-1$, *for some* $0 \leq t \leq s$ *with* $\alpha_t > 0$.
3. $max\{e|(\frac{J+I}{I})_e \neq 0\} = \alpha_s d_s + (n-1)(d_s-1)-1$.

Proof. 1. Firstly, we prove that $\frac{J+I}{I} \subset Soc(S/I)$. Since $Soc(S/I) = (O :_{S/I} \mathbf{m})$, it is enough to show that $\mathbf{m}J \subset I$.

We have $J = \sum_{t=0,\ \alpha_t>0}^s J_t$, where $J_t = (x_1\cdots x_n)^{d_t-1}(\mathbf{m}^{[d_t]})^{\alpha_t-1}\prod_{j>t}(\mathbf{m}^{[d_j]})^{\alpha_j}$. It is enough to prove that $x_iJ_t \subset I$ for any i and any t. Suppose $i=1$:

$$x_1J_t = x_1^{d_t}(x_2\cdots x_n)^{d_t-1}(\mathbf{m}^{[d_t]})^{\alpha_t-1}\prod_{j>t}(\mathbf{m}^{[d_j]})^{\alpha_j} \subset (x_2\cdots x_n)^{d_t-1}\prod_{j\geq t}(\mathbf{m}^{[d_j]})^{\alpha_j}.$$

On the other hand, $(x_2\cdots x_n)^{d_t-1} \in \prod_{j<t}(\mathbf{m}^{[d_j]})^{\alpha_j}$, because $d_t - 1 \geq \sum_{j<t}\alpha_j d_j$. Thus $x_1J_t \subset I$.

For the converse, we apply induction on α. If $\alpha = 1$ then $s = 0$ and $I = (x_1,\ldots,x_n) = \mathbf{m}$. $J = (x_1,\ldots,x_n)^{d_0-1} = S$, and obvious $Soc(S/I) = Soc(S/\mathbf{m}) = S/\mathbf{m}$. Let us suppose that $\alpha > 1$. We prove that if $w \in S\setminus I$ is a monomial such that $\mathbf{m}w \subset I$, then $w \in J$. Let $t_e = max\{t :\ x_e^{d_t-1}|w\}$. Renumbering $x_1,\ldots,x_n$ which does not affect either I or J, we may suppose that $t_1 \geq t_2 \geq \cdots \geq t_n$. We have two cases: (i)$t_1 > t_n$ and (ii)$t_1 = t_n$. But first, let's make the following remark: $(*)$ If $u = x_1^{\beta_1}\cdots x_n^{\beta_n} \in \prod_{j\geq t}\mathbf{m}^{d_j}$ and $\beta_i < d_t$ for certain i then $u/x_i^{\beta_i} \in \prod_{j\geq t}\mathbf{m}^{[d_j]}$ (the proof is similarly to [23, Lemma 3.5]).

In the case (i), there exists an index e such that $t_e > t_{e+1} = \cdots = t_n$. Then we have $w = (x_n\cdots x_{e+1})^{d_{t_n}-1}\cdot x_e^{d_{t_e}-1}\cdot y$, for a monomial $y \in S$. We consider two cases (a) x_e does not divide y and (b) x_e divide y. (a) From $x_nw = x_n^{d_{t_n}}\cdot(x_{n-1}\cdots x_{e+1})^{d_{t_n}-1}x_e^{d_{t_e}-1}\cdot y \in I$ we see that $y \in \prod_{j\geq t_e}(\mathbf{m}^{[d_j]})^{\alpha_j}$, by $(*)$. Therefore $w \in I$, because $x_e^{d_{t_e}-1} \in \prod_{j<t_e}(\mathbf{m}^{[d_j]})^{\alpha_j}$, which is an contradiction.

(b) In this case, $w = (x_n\cdots x_{e+1})^{d_{t_n}-1}x_e^{d_{t_e}}y'$, where $y' = y/x_e$. We claim that there exist $\lambda \leq t_e$ such that $\alpha_\lambda \neq 0$. Indeed, if all $\alpha_\lambda = 0$ for $\lambda \leq t_e$, then $I = \prod_{j=t_e+1}^s(\mathbf{m}^{[d_j]})^{\alpha_j}$ and $x_nw \in I$ implies $y' \in I$ because of the maximality of t_n and $(*)$. It follows $w \in I$, which is

false. Choose $\lambda \leq t_e$ maximal possible with $\alpha_\lambda \neq 0$. Set $w' = w/x_e^{d_\lambda}$. Note that $\mathbf{m}w \subset I$ implies

$$\mathbf{m}w \subset I' = (\mathbf{m}^{[d_{t_\lambda}]})^{\alpha_\lambda - 1} \prod_{j\neq\lambda} (\mathbf{m}^{[d_{t_j}]})^{\alpha_j}.$$

It is obvious that $x_q w' \in I'$ for $q \neq e$. Also, since $x_e^{d_{t_e+1}}$ does not divide $x_e w$ implies $x_e w' \in I'$. Choosing $\alpha' = \alpha - d_\lambda$, we get $\alpha'_j = \alpha_j$ for $j \neq \lambda$ and $\alpha'_\lambda = \alpha_\lambda - 1$ and therefore we can apply our induction hypothesis for I' (because $\alpha' < \alpha$) and for the ideal J' associated to I', which has the form:

$$J' = \sum_{q=0,\alpha'_q\neq 0} (x_1 \cdots x_n)^{d_q - 1} (\mathbf{m}^{[d_q]})^{\alpha'_q - 1} \prod_{j>q} (\mathbf{m}^{[d_j]})^{\alpha'_j},$$

and so $w = x_e^{d_\lambda} w' \in x_e^{d_\lambda} J' \subset J$.

It remains to consider the case (ii) in which we have in fact $t_1 = t_2 = \cdots = t_n$. If $y = w/(x_1 \cdots x_n)^{d_{t_n}-1} \in \mathbf{m}$, then there exists e such that $x_e|y$, and we apply our induction hypothesis as in the case (b) above. Thus we may suppose $y = 1$, i.e. $w = (x_1 \cdots x_n)^{d_{t_n}-1}$. Since $\mathbf{m}w \subset I$, we see that $\alpha_j = 0$ for $j > t_n$ and $\alpha_{t_n} = 1$ (otherwise $w \in I$, which is absurd). Thus $w \in J$.

2. Let $v = x_1^{q_t - 1}(x_2 \cdots x_n)^{d_t - 1}$. Then $deg(v) = q_t + (n-1)(d_t - 1) - 1$. But $v \in J$ and $v \notin I$, therefore $v \neq 0$ in $Soc(S/I) = \frac{J+I}{I}$.

3. Let $e_t = q_t + (n-1)(d_t - 1) - 1$ for $0 \leq t \leq s$. Let $t < s$. Then

$$e_{t+1} - e_t = q_{t+1} - q_t + (n-1)(d_{t+1} - d_t) = -\alpha_t d_t + (n-1)(d_{t+1} - d_t) \geq d_{t+1} - (\alpha_t + 1) d_t \geq 0, \text{ so}$$

$$max\{e|((J+I)/I)_e \neq 0\} = e_s = \alpha_s d_s + (n-1)(d_s - 1) - 1.$$

□

Remark 1.5.2. From the proof of the above lemma, we may easily conclude that for $n \geq 3$, $e_t = e_{t'}$ if and only if $t = t'$, and if $n = 2$, then $e_t = e_{t'}$ $(t < t')$ if and only if $\alpha_{t'-1} = d_{t'}/d_{t'-1}, \ldots, \alpha_t = d_{t+1}/d_t$.

Corollary 1.5.3. *With the notations of previous lemma and remark, let $0 \leq t \leq s$ be an integer such that $\alpha_t \neq 0$. Let $h_t = dim_K((I + J_t)/I)$. Then:*

1. *$G(J_t) \cap (I + J_{t'}) = 0$ for $0 \leq t' \leq s$, $t' \neq t$.*

2. *$h_t = \binom{n+\alpha_t-2}{n-1} \prod_{j>t} \binom{n+\alpha_j-1}{n-1}$.*

3. *$dim_K(Soc(S/I)_e) = \begin{cases} h_q, & if\ n \geq 3\ and\ e = e_q\ for\ a\ q \leq s\ with\ \alpha_q \neq 0. \\ \sum_q h_q, & if\ n = 2\ and\ q \in \{\epsilon | e = e_\epsilon\ for\ \epsilon \leq s\ with\ \alpha_\epsilon \neq 0\}. \\ 0, & otherwise. \end{cases}$.*

Proof. 1. First suppose $t' < t$. A minimal generator $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$ of J_t has the form

$$(x_1 \cdots x_n)^{d_t - 1} \prod_{j \geq t} (x_1^{\lambda_{1j} d_j} \cdots x_n^{\lambda_{nj} d_j}), \ where \ \sum_{\nu=1}^{n} \lambda_{\nu j} = \begin{cases} \alpha_j, & if \ j > t, \\ \alpha_t - 1, & if \ j = t. \end{cases}.$$

Thus, $\beta_i = d_t - 1 + \sum_{j=t}^{s} \lambda_{ij} d_j$. On the other hand, $d_t - 1 = \sum_{j=0}^{t-1} (d_{j+1}/d_j - 1) d_j$, so β_i has the writing $\sum_{j=0}^{s} \beta_{ij} d_j$, where $\beta_{ij} = d_{j+1}/d_j - 1$ for $j < t$ and $\beta_{ij} = \lambda_{ij}$ for $j \geq t$.

Assume that $x^\beta \in I + J_{t'}$ for a certain $t' < t$. Then there exists $\gamma \in \mathbb{N}^n$ such that $x^\gamma \in G(I)$ (or $x^\gamma \in G(J_{t'})$) and $x^\gamma | x^\beta$, that is $\gamma_i \leq \beta_i$ for all $1 \leq i \leq n$. Let $\gamma_i = \sum_{j=0}^{s} \gamma_{ij} d_j$, the $\mathbf{d}-$ decomposition of γ_i. We notice that $(\beta_{is}, \ldots, \beta_{i0}) \geq (\gamma_{is}, \ldots, \gamma_{i0})$ in the lexicographic order.

Note that all minimal generators x^γ of I have the same degree $\alpha < e_t$ and $\sum_{i=1}^{n} \gamma_{iq} = \alpha_q$ for each $0 \leq q \leq s$. Also all minimal generators x^γ of $J_{t'}$ have the same degree $e_{t'} < e_t$ and $\sum_{i=1}^{n} \gamma_{iq} = \alpha_q$ for each $t \leq q \leq s$. It follows $deg(x^\beta) > deg(x^\gamma)$ and so $\beta_i > \gamma_i$ for some i. Choose a maximal $q < s$ such that $\beta_{iq} > \gamma_{iq}$ for some i. Thus $\beta_{ij} = \gamma_{ij}$ for $j > q$. It follows $\beta_{iq} \geq \gamma_{iq}$ since $(\beta_{is}, \ldots, \beta_{i0}) \geq_{lex} (\gamma_{is}, \ldots, \gamma_{i0})$. If $q \leq t$ then we have

$$\alpha_q = \sum_{i=1}^{n} \gamma_{iq} < \sum_{i=1}^{n} \beta_{iq} = \sum_{i=1}^{n} \lambda_{iq} \leq \alpha_q,$$

which is not possible. It follows $q < t$ and so $\beta_{it} = \gamma_{it}$ for each i. But this is not possible because we get $\alpha_t = \sum_{i=1}^{n} \gamma_{it} = \sum_{i=1}^{n} \lambda_{it} = \alpha_t - 1$. Hence $x^\beta \notin I + J_{t'}$.

Suppose now $t' > t$. If $e_{t'} > e_t$, then $G(J_t) \cap G(J_{t'}) = \emptyset$ by degree reason. Assume $e_t = e_{t'}$. If follows $n = 2$ by the previous remark. If $x_1^{\beta_1} x_2^{\beta_2} \in G(J_t) \cap J_{t'}$ we necessarily get $x_1^{\beta_1} x_2^{\beta_2} \in G(J_t) \cap G(J_{t'})$ again by degree reason. But this is not possible since it implies that $\alpha_{t'} - 1 = \beta_{1t'} + \beta_{2t'} = \alpha_{t'}$.

2. and 3. follows from 1. □

Theorem 1.5.4. *Let $u = \prod_{q=1}^{r} x_{i_1}^{\alpha_q}$, where $2 \leq i_1 < i_2 < \cdots < i_r \leq n$. Let*

$$I = < u >_{\mathbf{d}} = \prod_{q=1}^{r} \prod_{j=0}^{s} (\mathbf{m}_q^{[d_j]})^{\alpha_{qj}},$$

where $\alpha_q = \sum_{j=0}^{s} \alpha_{qj} d_j$. Suppose $i_r = n$. Let $1 \leq a \leq r$ be an integer and

$$P_a(I) := \{(\lambda, t) \in \mathbb{N}^a \times \mathbb{N}^a | \ 1 \leq \lambda_1 < \cdots < \lambda_a = r, t_a > \cdots > t_1, \alpha_{\lambda_\nu t_\nu} \neq 0, \ for \ 1 \leq \nu \leq a\}.$$

Let $J = \sum_{a=1}^{r} \sum_{(\lambda,t) \in P_a(I)} J_{(\lambda,t)}$, where $J_{(\lambda,t)}$ is the ideal

$$\prod_{e=1}^{a} (x_{i_{\lambda_e}} \cdots x_{i_{\lambda_{e-1}}+1})^{d_{t_e} - 1} \prod_{\nu=1}^{a} \mathbf{m}_{\lambda_\nu}^{[d_{t_\nu}+1]} \prod_{j > t_\nu} (\mathbf{m}_{\lambda_\nu}^{[d_j]})^{\alpha_{\lambda_\nu j}} (\mathbf{m}_{\lambda_\nu}^{[d_{t_\nu}]})^{\alpha_{\lambda_\nu, t_\nu} - 1} \prod_{q=\lambda_{\nu-1}+1}^{\lambda_\nu - 1} \prod_{j \geq t_\nu} (m_q^{[d_j]})^{\alpha_{qj}},$$

where we denote $\mathbf{m}^{[d_{t_{a+1}}]} = S$. Then $Soc(S/I) = (J + I)/I$.

Proof. The proof will be given by induction on r, the case $r = 1$ being done in Lemma 1.5.1. Suppose that $r > 1$. For $1 \leq q \leq r$, let: $I_q = \prod_{e=1}^{q} \prod_{j=0}^{s} (\mathbf{m}_e^{[d_j]})^{\alpha_{ej}}$ and $S_q = k[x_1, x_2, \ldots, x_{i_q}]$ For t with $\alpha_{rt} \neq 0$, denote:

$$I^{(t)} = \mathbf{m}_{r-1}^{[d_t]} \prod_{j<t} (\mathbf{m}_{r-1}^{[d_j]})^{\alpha_{r-1,j}} I_{r-2}.$$

Let $J^{(t)}$ be an ideal in S_{r-1} such that $Soc(S_{r-1}/I^{(t)}) = (J^{(t)} + I^{(t)})/I^{(t)}$. The induction step is given in the following lemma:

Lemma 1.5.5. *Suppose $i_r = n$ and let*

$$J = \sum_{t=0,\alpha_{rt}\neq 0} (x_n \cdots x_{i_{r-1}+1})^{d_t - 1} \prod_{j>t} (\mathbf{m}_r^{[d_j]})^{\alpha_{rj}} \prod_{j\geq t} (\mathbf{m}_{r-1}^{[d_j]})^{\alpha_{r-1,j}} (\mathbf{m}_r^{[d_t]})^{\alpha_{rt}-1} J^{(t)}.$$

Then $Soc(S/I) = (J + I)/I$.

Proof. Let $w \in S \setminus I$ be a monomial such that $\mathbf{m}_r w \subset I$. As in the proof of lemma 1.5.1, we choose for each $1 \leq \rho \leq n$, $e_\rho = max\{e : \ x_\rho^{d_e - 1} | w\}$. Renumbering variables $\{x_n, \ldots, x_{i_{r-1}+1}\}$ (it does not affect I, J and $I^{(t)}$), we may suppose $e_n \leq e_{n-1} \leq \cdots \leq e_{i_{r-1}+1}$. Set $t = e_n$. We claim that $\alpha_{rt} \neq 0$. Indeed, if $\alpha_{rt} = 0$ then from $x_n w \in I$ we get $x_n w / x_n^{d_t - 1} \in \widetilde{I} = \prod_{j>t} (\mathbf{m}_r^{[d_j]})^{\alpha_{rj}} I_{r-1}$ because $x_n^{d_t-1} \in \prod_{j<t} (\mathbf{m}_r^{[d_j]})^{\alpha_{rj}}$. Since $t = e_n$ is maximal chosen, we get $w / x_n^{d_t - 1} \in \widetilde{I}$ and so $w \in I$ a contradiction.

Reduction to the case that $x_n^{d_t}$ does not divide w. Suppose that $w = x_n^{d_t} \widetilde{w}$ and set

$$\widetilde{I} = (\mathbf{m}_r^{[d_t]})^{\alpha_{rt}-1} \prod_{\epsilon \leq 0, \epsilon \neq t} (\mathbf{m}_r^{[d_\epsilon]})^{\alpha_{r\epsilon}} I_{r-1}.$$

We see that $\mathbf{m} w \in I \Leftrightarrow \mathbf{m} \widetilde{w} \in \widetilde{I}$. Replacing w and I with $\widetilde{w}$ and $\widetilde{I}$, we reduce our problem to a new $\widetilde{t} < t$. The above argument implies that $\widetilde{\alpha}_{r\widetilde{t}} \neq 0$, where $\widetilde{\alpha}$ is the 'new' α of $\widetilde{I}$.

Reduction to the case when $\alpha_{rj} = \alpha_{r-1,j} = 0$ for $j > t$, $\alpha_{rt} = 1$ and $\alpha_{r-1,t} = 0$. From $x_n w \in I$, we see that there exists $\rho < n$ such that $x_\rho^{d_j} | w$ for $j > t$ if $\alpha_{rj} \neq 0$, or $j = t$ if $\alpha_{rt} > 1$. Choose such maximal possible ρ. Set $w' = w / x_\rho^{d_j}$,

$$I' = (\mathbf{m}_r^{[d_j]})^{\alpha_{rj}-1} \prod_{\epsilon \geq 0, \epsilon \neq j} (\mathbf{m}_r^{[d_\epsilon]})^{\alpha_{r\epsilon}} I_{r-1}.$$

We see that $\mathbf{m} w \subset I \Leftrightarrow \mathbf{m} w' \subset I'$, because from $x_n w \in I$, we get $x_n w' \in I'$ from the maximality of ρ.

Let $\alpha'_{rj} = \alpha_{rj} - 1$ and $\alpha'_{q\epsilon} = \alpha_{q\epsilon}$ for $(q, \epsilon) \neq (r, j)$. α' is the 'new' α for I'. If we show that

$$w' \in J' = \sum_{e \geq 0, \alpha'_{re} \neq 0} (x_n \cdots x_{i_{r-1}+1})^{d_e - 1} \prod_{\epsilon > e} (\mathbf{m}_r^{[d_\epsilon]})^{\alpha_{r\epsilon}} \prod_{j \geq e} (\mathbf{m}_{r-1}^{[d_j]})^{\alpha_{r-1,j}} (\mathbf{m}_r^{[d_\epsilon]})^{\alpha_{re}-1} J^{(t)},$$

then $w = x_\rho^{d_j} w' \in \mathbf{m}_r^{[d_j]} J' \subset J$. Using this procedure, by recurrence we arrive to the case $\alpha_{rj} = 0$ for $j > t$ and $\alpha_{rt} = 1$. Again from $x_n w \in I$, we note that there exists $\rho < i_{r-1}$ such that $x_\rho^{d_j} | w$ for $j \geq t$ with $\alpha_{r-1,j} \neq 0$. Choose such maximal possible ρ and note that $\mathbf{m} w \subset I$ if and only if $\mathbf{m} w'' \in I''$ for $w'' = w / x_\rho^{d_j}$, where

$$I'' = (\mathbf{m}_{r-1}^{d_j})^{\alpha_{r-1,j-1}} \prod_{\epsilon \geq 0,\, \epsilon \neq j} (\mathbf{m}_{r-1}^{[d_\epsilon]})^{\alpha_{r-1,\epsilon}} \prod_{\epsilon \geq 0} (\mathbf{m}_r^{[d_\epsilon]})^{\alpha_{r\epsilon}} I_{r-2}.$$

As above, we reduce our problem to I'' and the α'', which is the new α of I'', is given by $\alpha''_{r-1,j} = \alpha_{r-1,j-1}$, $\alpha''_{q\epsilon} = \alpha_{q\epsilon}$ for $(q, \epsilon) \neq (r-1, j)$. Using this procedure, by recurrence we end our reduction.

Case $\alpha_{rj} = \alpha_{r-1j} = 0$ for $j > t$, $\alpha_{rt} = 1$, $\alpha_{r-1t} = 0$ and $x_n^{d_t}$ does not divide w. Let express $w = (x_n \cdots x_{i_{r-1}+1})^{d_t - 1} y$. We will show that y does not depend on $\{x_n, \ldots, x_{i_{r-1}+1}\}$. Indeed, if $n = i_{r-1} + 1$ then there is nothing to show since $x_n^{d_t}$ does not divide w. Suppose that $n > i_{r-1} + 1$, then from $x_n w \in I$ we get $y \in I_{r-1}$ because $x_{n-1}^{d_t - 1} \in \prod_{j<t} (\mathbf{m}_r^{d_j})^{\alpha_{rj}}$ and the variables $x_n, \ldots, x_{i_{r-1}+1}$ are regular on $S/I_{r-1}S$. If $y = x_\eta y'$ for $\eta > i_{r-1}$, then as above $y' \in I_{r-1}$. Thus $w \in x_\eta^{d_t} x_\rho^{d_t - 1} y' \subset I$ for any $\rho \neq \eta, i_{r-1} < \rho \leq n$, a contradiction.

Note that $\mathbf{m}_r w \in I \Rightarrow \mathbf{m}_{r-1} y \in I^{(t)}$ and so $w \in (x_n \cdots x_{i_{r-1}+1})^{d_t - 1} J^{(t)}$. Since $\alpha_{rj} = \alpha_{r-1j} = 0$ for $j > t$ and $\alpha_{rt} = 1$ and $\alpha_{r-1,t} = 0$, we get $w \in J$. Conversely, if $y \in J^{(t)}$, then it is clear that $w \in J$. $\square$

We see by the above lemma that:

$$(*)\ J = \sum_{e \geq 0, \alpha_{re} \neq 0} (x_n \cdots x_{i_{r-1}+1})^{d_e - 1} \prod_{j > e} (\mathbf{m}_{r-1}^{[d_j]})^{\alpha_{rj}} \prod_{j \geq e} (\mathbf{m}_{r-1}^{[d_j]})^{\alpha_{r-1,j}} (\mathbf{m}_r^{[d_e]})^{\alpha_{re} - 1} J^{(e)}.$$

Since $\lambda_a = r - 1$, by the induction hypothesis applied to $I^{(e)}$ we get:

$$J^{(e)} = \sum_{a=1}^{r-1} [\sum_{(\lambda,t) \in P_a(I^{(e)}), t_a = e} \prod_{s=1}^{a} (x_{i_{\lambda_s}} \cdots x_{i_{\lambda_{s-1}+1}})^{d_{t_s} - 1} \cdot J'_{(\lambda,t)} +$$

$$+ \sum_{(\lambda,t) \in P_a(I^{(e)}), t_a < e} \prod_{s=1}^{a} (x_{i_{\lambda_s}} \cdots x_{i_{\lambda_{s-1}+1}})^{d_{t_s} - 1} \cdot J''_{(\lambda,t)}],\ \textit{where}$$

$$J'_{(\lambda,t)} = \prod_{q=\lambda_{a-1}+1}^{\lambda_a - 1} \prod_{j \geq e} (\mathbf{m}_q^{[d_j]})^{\alpha_{qj}} \widetilde{J}_{(\lambda,t)}\ \textit{and}$$

$$J''_{(\lambda,t)} = \mathbf{m}_{r-1}^{[d_e]} \prod_{j > t_a}^{e-1} (\mathbf{m}_{\lambda_a}^{[d_j]})^{\alpha_{\lambda_a,j}} (\mathbf{m}_{\lambda_a}^{[d_{t_a}]})^{\alpha_{\lambda_a,t_a-1}} \cdot \prod_{q=\lambda_{a-1}+1}^{\lambda_a - 1} \prod_{j \geq t_a} (\mathbf{m}_q^{[d_j]})^{\alpha_{qj}} \widetilde{J}_{(\lambda,t)},\ \textit{and}$$

$$\widetilde{J}_{(\lambda,t)} = \prod_{\nu=1}^{a-1} \mathbf{m}_{\lambda_\nu}^{[d_{t_{\nu+1}}]} \prod_{j > t_\nu} (\mathbf{m}_{\lambda_\nu}^{[d_j]})^{\alpha_{\lambda_\nu,j}} (\mathbf{m}_{\lambda_\nu}^{[d_{t_\nu}]})^{\alpha_{\lambda_\nu,t_\nu - 1}} \cdot \prod_{q=\lambda_{\nu-1}+1}^{\lambda_\nu - 1} \prod_{j \geq t_\nu} (\mathbf{m}_q^{[d_j]})^{\alpha_{qj}}.$$

If $t_a = e$, set $\lambda'_\nu = \lambda_\nu$ for $\nu < a$, $\lambda'_a = r$ and see that $(\lambda', t) \in P_a(I)$. If $t_a < e$, then put $\lambda''_\nu = \lambda_\nu$ for $\nu \leq a$, $\lambda''_{a+1} = r$, $t''_\nu = t_\nu$ for $\nu \leq a$ and $t''_{a+1} = e$ and then $(\lambda'', t) \in P_{a+1}(I)$. Substituting $J^{(e)}$ in $(*)$, we get the following expression for J:

$$\sum_{a=1}^{r-1} \sum_{(\lambda',t)\in P_a(I)} \prod_{\nu=1}^{a} (x_{i_{\lambda'_\nu}} \cdots x_{i_{\lambda'_{\nu-1}+1}})^{d_{t_\nu}-1} \cdot [\prod_{j>e} (\mathbf{m}_{\lambda'_a}^{[d_j]})^{\alpha_{\lambda'_a j}} (\mathbf{m}_{\lambda'_a}^{[d_e]})^{\alpha_{\lambda'_a e}-1} \cdot \prod_{q=\lambda'_{a-1}+1}^{\lambda'_a-1} \prod_{j\geq e} (\mathbf{m}_q^{[d_j]})^{\alpha_{qj}}]\cdot$$

$$\cdot \widetilde{J}_{(\lambda,t)} + \sum_{a=1}^{r-1} \sum_{(\lambda'',t'')\in P_{a+1}(I)} \prod_{\nu=1}^{a+1} (x_{i_{\lambda''_\nu}} \cdots x_{i_{\lambda''_{\nu-1}+1}})^{d_{t_\nu}-1} \cdot [\prod_{j>e} (\mathbf{m}_{\lambda''_{a+1}}^{[d_j]})^{\alpha_{\lambda''_{a+1},j}} (\mathbf{m}_{\lambda''_{a+1}}^{[d_{t''_{a+1}}]})^{\alpha_{\lambda''_{a+1} t''_{a+1}}-1}]$$

$$[\mathbf{m}_{\lambda''_a}^{[d_{t''_{a+1}}]} \prod_{j\geq t''_a} (\mathbf{m}_{\lambda''_a}^{[d_j]})^{\alpha_{\lambda''_{a+1},j}} (\mathbf{m}_{\lambda''_a}^{[d_{t''_a}]})^{\alpha_{\lambda''_a t''_a}-1} \prod_{q=\lambda''_{a-1}+1}^{\lambda''_a-1} \prod_{j\geq t''_a} (\mathbf{m}_q^{[d_j]})^{\alpha_{qj}}] \cdot \widetilde{J}_{(\lambda,t)}.$$

Since all the pairs of $P_b(I)$ have the form (λ', t) or (λ'', t'') for a pair $(\lambda, t) \in P_b(I)$ or $(\lambda, t) \in P_{b-1}(I)$ respectively, it is not hard to see that the expression above is the formula of J as stated. ☐

Let $s_q = max\{j|\alpha_{qj} \neq 0\}$, $d_{qt} = \sum_{e=1}^{q} \sum_{j\geq t}^{s_q} \alpha_{ej} d_j$, $D_q = d_{q,s_q} + (i_q - 1)(d_{s_q} - 1)$ for $1 \leq q \leq r$.

Corollary 1.5.6. *With the notation and hypothesis of above theorem, for $(\lambda, t) \in P_a(I)$ let:*

$$d_{(\lambda,t)} = \sum_{\nu=1}^{a} \sum_{q=\lambda_{\nu-1}+1}^{\lambda_\nu} \sum_{j\geq t_\nu} \alpha_{qj} d_j. \ Then:$$

1. *$Soc(I_{r-1}S/I) = Soc(S/I)$.*
2. *$((J + I)/I)_e \neq 0$, if and only if $e = d_{(\lambda,t)} + \sum_{\nu=1}^{a} (i_{\lambda_\nu} - i_{\lambda_{\nu-1}})(d_{t_\nu} - 1) - d_{t_1}$, for some $1 \leq a \leq r$ and $(\lambda, t) \in P_a(I)$.*
3. *$c = max\{e|((J + I)/I)_e \neq 0\} = d_{r,s_r} + (n - 1)(d_{s_r} - 1) - 1$.*

Proof. 1.Note that $J_{(\lambda,t)}$ is contained in

$$\prod_{q=1,q\notin\{\lambda_1,\ldots,\lambda_q\}}^{r} (\mathbf{m}_q^{[d_j]})^{\alpha_{qj}} \prod_{\nu=1}^{a} [\prod_{j\neq t_\nu} (\mathbf{m}_{\lambda_\nu}^{d_j})^{\alpha_{\lambda_\nu,j}} (\mathbf{m}_{\lambda_\nu}^{t_\nu})^{\alpha_{\lambda_\nu,t_\nu}-1}] \prod_{\epsilon=1}^{a-1} \mathbf{m}_{\lambda_{\epsilon+1}}^{d_{t_\epsilon}+1}.$$

Since $\mathbf{m}_{\lambda_\epsilon}^{d_{t_\epsilon+1}} \subset \mathbf{m}_{\lambda_\epsilon}^{d_{t_\epsilon}}$ for $t_{\epsilon+1} > t_\epsilon$ and $\lambda_a = r$ if follows that

$$J \subset \prod_{j\neq t_a} (\mathbf{m}_r^{[d_j]})^{\alpha_{rj}} (\mathbf{m}_r^{[d_{t_a}]})^{\alpha_{rt_a}-1} I_{r-1},$$

as desired.

2.If $((J+I)/I)_e \neq 0$ then there exists a monomial $u \in J \setminus I$ of degree e. But $u \in J$, implies that there exists $a \in \{1,\ldots,r\}$ and $(\lambda,t) \in P_a(I)$ such that $u \in J_{(\lambda,t)}$. Thus the degree of u is $e = d_{(\lambda,t)} + \sum_{\nu=1}^{a}(i_{\lambda_\nu} - i_{\lambda_{\nu-1}})(d_{t_\nu}-1) - d_{t_1}$, as required.

Conversely, let $e = d_{(\lambda,t)} + \sum_{\nu=1}^{a}(i_{\lambda_\nu} - i_{\lambda_{\nu-1}})(d_{t_\nu}-1) - d_{t_1}$ for some $a \in \{1,\ldots,r\}$ and $(\lambda,t) \in P_a(I)$. We show that the monomial

$$w = \prod_{\nu=1}^{a}(x_{i_{\lambda_\nu}} \cdots x_{i_{\lambda_{\nu-1}+1}})^{d_{t_\nu}-1} \cdot x_1^{d_{(\lambda,t)}-d_{t_1}} \in J \setminus I.$$

Obvious $w \in J$. Let us assume that $w \notin I$. Then $w/x_{i_{\lambda_a}}^{d_{t_a}-1} \in \prod_{j\geq t_a}(\mathbf{m}_{\lambda_a}^{[d_j]})^{\alpha_{\lambda_a j}} I_{\lambda_a - 1}$ because $x_{i_{\lambda_a}}^{d_{t_a}-1} \in \prod_{j<t_a}(\mathbf{m}_{\lambda_a}^{[d_j]})^{\alpha_{\lambda_a j}}$ and $x_{i_{\lambda_a}} \notin \mathbf{m}_j$ for $j < \lambda_a$. Inductively we get that:

$$w/(x_{i_{\lambda_a}} \cdots x_{i_{\lambda_{a-1}+1}})^{d_{t_a}-1} \in \prod_{q=\lambda_{a-1}+1}^{\lambda_a} \prod_{j\geq t_a} (\mathbf{m}_{\lambda_a}^{[d_j]})^{\alpha_{qj}} I_{\lambda_a - 1}.$$

Following the same reduction and using that $t_a > \cdots > t_1$ we obtain that:

$$x_1^{d_{(\lambda,t)}-d_{t_1}} \in \prod_{\nu=1}^{a} \prod_{q=\lambda_{\nu-1}+1}^{\lambda_\nu} \prod_{j\geq t_\nu} (\mathbf{m}_{\lambda_a}^{[d_j]})^{\alpha_{qj}}.$$

So $d_{(\lambda,t)} - d_{t_1} \geq d_{(\lambda,t)}$, a contradiction.

3.Note that $c = d_{(\lambda',t')}$ for $(\lambda',t') \in P_1(I)$ with $\lambda' = \lambda_1 = r$ and $t' = t_1 = s_r$. We have to show that:

$$c = d_{r,s_r} + (n-1)(d_{s_r}-1) - 1 \leq d_{(\lambda,t)} + \sum_{\nu=1}^{a}(i_{\lambda_\nu} - i_{\lambda_{\nu-1}})(d_{t_\nu}-1) - d_{t_1},$$

for any $1 \leq a \leq r$ and $(\lambda,t) \in P_a(I)$. Since $d_{s_r} - 1 \leq (d_{t_\nu}-1) + \sum_{j\leq t_\nu}^{s_r-1} \alpha_{qj} d_j$ for all q with $i_{\nu-1} < q \leq i_\nu$, we see that:

$$d_{r,s_r} + (n-1)(d_{s_r}-1) - 1 \geq d_{(\lambda,t)} + \sum_{\nu=2}^{a}(i_{\lambda_\nu} - i_{\lambda_{\nu-1}})(d_{t_\nu}-1) + (i_{\lambda_1}-1)(d_{t_1}-1) - 1.$$

On the other hand, $(i_{\lambda_1} - i_{\lambda_0})(d_{t_1}-1) = (i_{\lambda_1}-1)(d_{t_1}-1) + d_{t_1} - 1$, and replacing that in the above relation we obtained what is required. □

Example 1.5.7. Let $\mathbf{d} : 1|2|4|12$.

1. Let $u = x_3^{21}$. We have $\alpha_0 = 1$, $\alpha_1 = 0$, $\alpha_2 = 2$ and $\alpha_3 = 1$ so:

   $$I = < u >_{\mathbf{d}} = (x_1,x_2,x_3)(x_1^4,x_2^4,x_3^4)^2(x_1^{12},x_2^{12},x_3^{12}).$$

   Let $J = \sum_{t=0,\alpha_t>0} J_t$, where $J_t = (x_1x_2x_3)^{d_t-1}(x_1^{d_t},x_2^{d_t},x_3^{d_t})^{\alpha_t-1}\prod_{j>t}(x_1^{d_j},x_2^{d_j},x_3^{d_j})^{\alpha_j}$.
   $J_0 = (x_1x_2x_3)^{1-1} \cdot (x_1,x_2,x_3)^{1-1} \cdot \prod_{j>t}(x_1^{d_j},x_2^{d_j},x_3^{d_j})^{\alpha_j} = (x_1^4,x_2^4,x_3^4)^2(x_1^{12},x_2^{12},x_3^{12})$.
   $J_2 = (x_1x_2x_3)^{4-1}(x_1^4,x_2^4,x_3^4)^{2-1}(x_1^{12},x_2^{12},x_3^{12}) = (x_1x_2x_3)^3(x_1^4,x_2^4,x_3^4)(x_1^4,x_2^4,x_3^{12})$ and
   $J_3 = (x_1x_2x_3)^{12-1} = (x_1x_2x_3)^{11}$. From 1.5.1 , $Soc(S/I) = (J+I)/I$.

2. Let $u = x_2^9 x_3^{16}$. We have $r = 2$, $i_1 = 2$ and $i_2 = 3$. Also $\alpha_{10} = 1$, $\alpha_{12} = 2$, $\alpha_{22} = 1$, $\alpha_{23} = 1$ and the other components of α are zero. Then

$$I =< u >_{\mathbf{d}} =< x_2^9 >_{\mathbf{d}} < x_3^{16} >_{\mathbf{d}} = (x_1, x_2)(x_1^4, x_2^4)^2(x_1^4, x_2^4, x_3^4)(x_1^{12}, x_2^{12}, x_3^{12}).$$

We have two possible partitions: (a) (2) and (b) $(1 < 2)$.

(a)$\lambda = \lambda_1 = 2$, $t = t_1$ such that $\alpha_{2t} \neq 0$. We have two possible t: $t = 2$ or $t = 3$.

(i)For $t = 2$ we obtain (according to the Theorem 1.5.4) the following part of the socle:

$$J_{(2,2)} = (x_1x_2x_3)^3(x_1^{12}, x_2^{12}, x_3^{12})(x_1^4, x_2^4)^4$$

(ii)For $t = 3$ we obtain:

$$J_{(2,3)} = (x_1x_2x_3)^{11}$$

(b)$1 = \lambda_1 < \lambda_2 = 2$, $t = (t_1, t_2)$ such that $\alpha_{\lambda_e, t_e} \neq 0$ for $1 \leq e \leq 2$ and $t_1 < t_2$. According to our expressions for α_i we have three possible cases: $t_1 = 0, t_2 = 2$ or $t_1 = 0, t_2 = 3$ or $t_1 = 2, t_2 = 3$.

(i)For $t_1 = 0$ and $t_2 = 2$ we obtain:

$$J_{(1,2),(0,2)} = x_3^3(x_1^4, x_2^4)(x_1^4, x_2^4)^2(x_1^{12}, x_2^{12}, x_3^{12}).$$

(ii)For $t_1 = 0$ and $t_2 = 3$ we obtain:

$$J_{(1,2),(0,3)} = x_3^{11}(x_1^{12}, x_2^{12})(x_1^4, x_2^4)^2$$

(iii)For $t_1 = 2$ and $t_2 = 3$ we obtain:

$$J_{(1,2),(2,3)} = x_1^3x_2^3x_3^{11}(x_1^{12}, x_2^{12})(x_1^4, x_2^4)$$

From 2.4 it follows that if $J = J_{(2,2)} + J_{(2,3)} + J_{(1,2),(0,2)} + J_{(1,2),(0,3)} + J_{(1,2),(2,3)}$ then $Soc(S/I) = (I + J)/J$.

1.6 A generalization of Pardue's formula.

In this section, we give a generalization of a theorem proved by Aramova-Herzog [3] and Herzog-Popescu [24] which is known as "Pardue's formula".

Let $1 \leq i_1 < i_2 < \cdots < i_r = n$ and let $\alpha_1, \ldots, \alpha_r$ be some positive integers. Let $u = \prod_{i=1}^r x_{i_q}^{\alpha_q} \in S = K[x_1, \ldots, x_n]$. Let $I =< u >_{\mathbf{d}}$ the principal $\mathbf{d}$-fixed ideal generated by u. From Proposition 1.4.11 it follows that $I = \prod_{r=1}^q \prod_{j=0}^s (\mathbf{m}_q^{[d_j]})^{\alpha_{qj}}$, where $\alpha_q = \sum_{j=0}^s \alpha_{qj} d_j$. If $i_1 = 1$, it follows that $I = x_1^{\alpha_1} I'$, where $I' = \prod_{r=2}^q \prod_{j=0}^s (\mathbf{m}_q^{[d_j]})^{\alpha_{qj}}$, and therefore $reg(I) = \alpha_1 + reg(I')$. Thus, we may assume $i_1 \geq 2$.

If N is a graded S-module of finite length, we denote $s(N) = max\{i | N_i \neq 0\}$. Let $s_q = max\{j | \alpha_{qj} \neq 0\}$ and $d_{qt} = \sum_{e=1}^q \sum_{j \geq t}^{s_e} \alpha_{ej} d_j$. Let $D_q = d_{qs_q} + (i_q - 1)(d_{s_q} - 1)$, for $1 \leq q \leq r$. With this notations we have:

Theorem 1.6.1. *$reg(I) = max_{1\leq q\leq r} D_q$. In particular, if $I =< x_n^\alpha >_{\mathbf{d}}$ and $\alpha = \sum_{t=0}^{s} \alpha_t d_t$ with $\alpha_s \neq 0$ then $reg(I) = \alpha_s d_s + (n-1)(d_s - 1)$.*

Proof. Let $I_\ell = \prod_{q=1}^{r-\ell} \prod_{j=0}^{s} (\mathbf{m}_q^{[d_j]})^{\alpha_{qj}}$, for $0 \leq \ell \leq r$. Then $I = I_0 \subset I_1 \subset \cdots \subset I_r = S$ is the sequential chain of ideals of I, i.e. $I_{\ell+1} = (I_\ell : x_{n_\ell}^\infty)$, where $n_\ell = i_{r-\ell}$. Let $S_\ell = k[x_1, \ldots, x_{n_\ell}]$ and $m_\ell = (x_1, \ldots, x_{n_\ell})$. Let $J_\ell \subset S_\ell$ be the ideal generated by $G(I_\ell)$.

The Corollary 1.5.6 implies that $c_e = D_e - 1$ is the maximal degree for a nonzero element of $Soc(S_\ell/J_\ell)$. Proposition 1.2.2 implies $reg(I) = max\{s(J_\ell^{sat}/J_\ell) \mid \ell = 0, \ldots, r-1)\} + 1$. Also, from the Corollary 1.5.6, we get

$$s(J_\ell^{sat}/J_\ell) = s(Soc(J_\ell^{sat}/J_\ell)) = s(Soc(S_\ell/J_\ell)) = D_e - 1,$$

which complete the proof. Indeed, for the first equality, if $u \in J_\ell^{sat} \setminus J_\ell$ and $deg(u) = s(J_\ell^{sat}/J_\ell)$ it follows that $u \in Soc(J_\ell^{sat}/J_\ell)$, since $\mathbf{m}u \subset J_\ell$ by degree reasons. □

Corollary 1.6.2. *$reg(I) \leq n \cdot (deg(u) - 1) + 1$.*

Corollary 1.6.3. *S/I has at most r-corners among $(i_q, D_q - 1)$ for $1 \leq q \leq r$. If $i_1 = 1$ we replace $(i_1, D_1 - 1)$ with $(1, \alpha_1)$. The corresponding extremal Betti numbers are $\beta_{i_q, D_q + i_q - 1}$.*

Proof. By Theorem 1.1.6 combined with the proof of Theorem 1.6.1, S/I has at most r-corners among $(n_\ell, s(I_{\ell+1}S_\ell/I_\ell S_\ell))$ and is enough to apply Corollary 1.5.6. □

Example 1.6.4. Let $\mathbf{d} : 1|2|4|12$.

1. Let $u = x_3^{21} \in k[x_1, x_2, x_3]$. We have $21 = 1 \cdot 1 + 0 \cdot 2 + 2 \cdot 4 + 1 \cdot 12$. From 3.1, we get:

$$reg(< u >_{\mathbf{d}}) = 1 \cdot 12 + (3-1) \cdot (12-1) = 34.$$

2. Let $u = x_1^2 x_2^{16} x_3^9$. Then $reg(< u >_{\mathbf{d}}) = 2 + reg(< u' >_{\mathbf{d}})$, where $u' = u/x_1^2$. We compute $reg(< u' >_{\mathbf{d}})$. With the notations above, we have $i_1 = 2$, $i_2 = 3$, $r = 2$, $\alpha_1 = 16$ and $\alpha_2 = 9$. We have $\alpha_1 = 1 \cdot 4 + 1 \cdot 12$ and $\alpha_2 = 1 \cdot 1 + 2 \cdot 4$, thus $s_1 = 3$ and $s_2 = 2$. $D_1 = d_{13} + (2-1)(d_3 - 1) = 12 + 11 = 23$ and $D_2 = d_{22} + (3-1)(d_2 - 1) = 24 + 6 = 30$. In conclusion, $reg(< u >_{\mathbf{d}}) = 2 + max\{23, 30\} = 32$.

3. Let $u = x_a^\alpha x_b^\beta$, with $1 < a < b \leq n$ and $\beta < \alpha$, $\beta|\alpha$ be two integers. Set $d_1 = \beta$, $d_2 = \alpha$ and let I be the principal $\mathbf{d}$-fixed ideal generated by u. Obviously $I = SBT(u) = (x_1^\alpha, \ldots, x_a^\alpha)(x_1^\beta, \ldots, x_b^\beta)$. We have $i_1 = a$, $i_2 = b$, $s_1 = 2$, $s_2 = 1$, $d_{1s_1} = \alpha$, $d_{2s_2} = \alpha + \beta$, $D_1 = \alpha + (a-1)(\alpha - 1) = a(\alpha - 1) + 1 = \chi_1 + 1$, $D_2 = \alpha + \beta + (b-1)(\beta - 1) = \alpha + b(\beta - 1) + 1 = \chi_2 + 1$ in the notations of Theorem 1.3.6. Note that 1.3.6 and 1.6.1 give the same regularity of I, namely $max\{D_1, D_2\}$.

Definition 1.6.5. *We say that a monomial ideal $I \subset S$ is a $\mathcal{D}$-fixed ideal , if I is a sum of **d**-fixed ideals, for various **d**-sequences.*

Since any **d**-fixed ideal is a Borel type ideal and a sum of Borel type ideals is stil a Borel type ideal, it follows that any $\mathcal{D}$-fixed ideal I is a Borel type ideal. Therefore, from Corollary 1.2.11 we get the next:

Corollary 1.6.6. *If I is a $\mathcal{D}$-fixed ideal then $reg(I) = min\{e: \ e \geq deg(I), I_{\geq e}$ is stable$\}$.*

We mention that this result was first obtained as a consequence of the proof of Pardue's formula by Herzog-Popescu [24] in the special case of a principal p-Borel ideal.

1.7 d-fixed ideals generated by powers of variables.

Firstly, let fix some notations. Let $u_1, \dots, u_m \in S$ be some monomials. We say that I is the **d**-fixed ideal generated by $u_1, \dots, u_m$, if I is the smallest **d**-fixed ideal , w.r.t inclusion, which contained $u_1, \dots, u_m$, and we write $I =< u_1, \dots, u_m >_{\mathbf{d}}$. In particular, if $m = 1$, we say that I is the principal **d**-fixed ideal generated by $u = u_1$ and we write $I =< u >_{\mathbf{d}}$.

Lemma 1.7.1. *If $1 \leq j \leq j' \leq n$ and $\alpha \geq \beta$ are positive integer, then $< x_j^{\alpha} > \subset < x_{j'}^{\beta} >$.*

Proof. Indeed, using Proposition 1.4.9 it is enough to notice that $< x_j^{\alpha} > \subset < x_{j'}^{\alpha} >$ which is true because $x_j^{\alpha} \in < x_{j'}^{\alpha} >$. □

Our next goal is to give the minimal set of generators for a **d**-fixed ideal generated by some powers of variables. Using the previous lemma, we had reduced to the next case:

Proposition 1.7.2. *Let $n \geq 2$ and let $1 \leq i_1 < i_2 < \cdots < i_r = n$ be some integers. Let $\alpha_1 < \alpha_2 < \cdots < \alpha_r$ some positive integers. Then*

$$I =< x_{i_1}^{\alpha_1}, x_{i_2}^{\alpha_2}, \dots, x_{i_r}^{\alpha_r} >_{\mathbf{d}} = \sum_{q=1}^{r} I^{(q)}, \text{with } I^{(q)} = \sum_{\substack{0 \leq \gamma_1, \dots, \gamma_q \leq_{\mathbf{d}} \alpha_q, \\ \gamma_1 + \cdots + \gamma_i < \alpha_i, \ for\ i < q \\ \gamma_1 + \cdots + \gamma_i <_d \alpha_q, \ for\ i < q \\ \gamma_1 + \cdots + \gamma_q = \alpha_q}} \prod_{e=1}^{q} \prod_{t=0}^{s} (\mathbf{n}_e^{[d_t]})^{\gamma_{et}},$$

where $\mathbf{n}_e = \{x_{i_{e-1}+1}, \dots, x_{i_e}\}$, $\mathbf{n}_e^{[d_t]} = \{x_{i_{e-1}+1}^{d_t}, \dots, x_{i_e}^{d_t}\}$, $i_0 = 0$ and $\gamma_e = \sum_{t=0}^{s} \gamma_{et}$.

Proof. Let $\mathbf{m}_q = \{x_1, \dots, x_{i_q}\}$ for $1 \leq q \leq r$. Obviously, $\mathbf{n}_q = \mathbf{m}_q \setminus \mathbf{m}_{q-1}$ for $q > 0$ and $\mathbf{n}_1 = \mathbf{m}_1$. Using the obvious fact that I is the sum of principal **d**-fixed ideals generated by the **d**-generators of I together with Proposition 1.4.8 we get:

$$I = \sum_{q=1}^{r} \prod_{t=0}^{s} (\mathbf{m}_q^{[d_t]})^{\alpha_{qt}}, \ where\ \alpha_q = \sum_{t=0}^{s} \alpha_{qt} d_t.$$

Denote $S_q = K[x_1, \dots, x_{i_q}]$ for $1 \leq q \leq r$. In order to obtain the required formula, we use induction on $r \geq 1$, the case $r = 1$ being obvious. Let $r > 1$ and assume that the assertion is true for $r - 1$, i.e

$$I' =< x_{i_1}^{\alpha_1}, \dots, x_{i_{r-1}}^{\alpha_{r-1}} >_{\mathbf{d}} = \sum_{q=1}^{r-1} \sum_{\substack{0 \leq \gamma_1, \dots, \gamma_q \leq_{\mathbf{d}} \alpha_q, \\ \gamma_1 + \cdots + \gamma_i < \alpha_i, \ for\ i < q \\ \gamma_1 + \cdots + \gamma_i <_d \alpha_q, \ for\ i < q \\ \gamma_1 + \cdots + \gamma_q = \alpha_q}} \prod_{e=1}^{q} \prod_{t=0}^{s} (\mathbf{n}_e^{[d_t]})^{\gamma_{et}} \subset S_{r-1}.$$

Obviously, $I = I'S+ < x_n^{\alpha_r} >_{\mathbf{d}} = I'S + \prod_{t=0}^{s}(\mathbf{m}_r^{[d_t]})^{\alpha_{rt}}$. Also, $I'S$ and I' have the same set of minimal generators and none of the minimal generators of $I'S$ is in $I^{(r)}$, because of degree.

But, a minimal generator of $< x_n^{\alpha_r} >_{\mathbf{d}}$ is of the form $w = \prod_{t=0}^{s}\prod_{j=1}^{n} x_j^{\lambda_{tj}d_t}$ with $0 \leq \lambda_{tj}$ and $\sum_{j=1}^{n}\lambda_{tj} = \alpha_{rt}$. Suppose $w \notin I'S$. In order to complete the proof, we will show that $w \in I^{(r)}$. Let $v_q = \prod_{t=0}^{s}\prod_{j=i_{q-1}+1}^{i_q} x_j^{\lambda_{tj}d_t}$ and let $w_q = \prod_{e=1}^{q} v_e$. Obvious, $w = v_1 \cdots v_r = w_r$. Since $w \notin I'$ it follows that $w_q \notin I^{(q)}$ for any $1 \leq q \leq r-1$. But $w_q \notin I^{(q)}$ implies $(*)$ $\sum_{t=0}^{s}\sum_{j=1}^{i_q}\lambda_{tj}d_t < \alpha_q$, otherwise $w_q \in < x_{i_q}^{\alpha_q} S_q >_{\mathbf{d}} S_{r-1} \subset I'$ and thus $w \in I'$, a contradiction. We choose $\gamma_e = \sum_{t=0}^{s}\sum_{j=i_{e-1}+1}^{i_e}\lambda_{tj}d_t$ for $1 \leq e \leq r$. For $1 \leq q < r$, $(*)$ implies $\gamma_1 + \cdots + \gamma_q < \alpha_q$. On the other hand, it is obvious that $\gamma_1 + \cdots + \gamma_e \leq_d \alpha_r$ for any $1 \leq e \leq r$ and $\gamma_1 + \cdots + \gamma_r = \alpha_r$. Thus $w \in I^{(r)}$ as required. $\square$

Example 1.7.3. Let $\mathbf{d} : 1|2|4|12$ and let $I =< x_2^7, x_3^{10}, x_5^{17} > \subset K[x_1, \ldots, x_5]$. We have $7 = 1\cdot 1 + 1\cdot 2 + 1 \cdot 4$, $10 = 1\cdot 2 + 2 \cdot 4$, $17 = 1 \cdot 1 + 1 \cdot 4 + 1 \cdot 12$. We have

$$I^{(1)} =< x_2^7 >_{\mathbf{d}} = (x_1, x_2)(x_1^2, x_2^2)(x_1^4, x_2^4).$$

In order to compute $I^{(2)}$, we need to find all the pairs (γ_1, γ_2) such that $\gamma_1 < 7$, $\gamma_1 <_{\mathbf{d}} 10$ and $\gamma_2 = 10 - \gamma_1$. We have 4 pairs, namely $(0, 10)$, $(2, 8)$, $(4, 6)$ and $(6, 4)$, thus

$$I^{(2)} = (x_1^2, x_2^2)(x_1^4, x_2^4)x_3^4 + (x_1^4, x_2^4)x_3^6 + (x_1^2, x_2^2)x_3^8 + (x_3^{10}).$$

In order to compute $I^{(3)}$, we need to find all $(\gamma_1, \gamma_2, \gamma_3)$ such that $\gamma_1 < 7$, $\gamma_1 + \gamma_2 < 10$, $\gamma_1 <_{\mathbf{d}} 17$, $\gamma_1 + \gamma_2 <_{\mathbf{d}} 17$ and $\gamma_3 = 17 - \gamma_1 + \gamma_2$. If $\gamma_1 = 0$ then, the pair (γ_2, γ_3) is one of the following:$(0, 17)$,$(1, 16)$,$(4, 13)$ or $(5, 12)$. If $\gamma_1 = 1$ then, the pair (γ_2, γ_3) is one of the following:$(0, 16)$ of $(4, 12)$. If $\gamma_1 = 4$ then, the pair (γ_2, γ_3) is one of the following:$(0, 13)$ of $(1, 12)$. If $\gamma_1 = 5$ then, the pair (γ_2, γ_3) is $(0, 12)$. Thus

$$I^{(3)} = (x_1, x_2)(x_1^4, x_2^4)(x_4^{12}, x_5^{12}) + (x_1^4, x_2^4)x_3(x_4^{12}, x_5^{12}) + (x_1^4, x_2^4)(x_4, x_5)(x_4^{12}, x_5^{12}) +$$
$$+ (x_1, x_2)x_3^4(x_4^{12}, x_5^{12}) + (x_1, x_2)(x_4^4, x_5^4)(x_4^{12}, x_5^{12}) + x_3(x_4^4, x_5^4)(x_4^{12}, x_5^{12}) +$$
$$+ x_3^4(x_4, x_5)(x_4^{12}, x_5^{12}) + x_3^5(x_4^{12}, x_5^{12}) + (x_4, x_5)(x_4^4, x_5^4)(x_4^{12}, x_5^{12}).$$

From Proposition 1.7.2, we get $I = I^{(1)} + I^{(2)} + I^{(3)}$.

Remark 1.7.4. For any $1 \leq q \leq r$ and any nonnegative integers $\gamma_1, \ldots, \gamma_q \leq_{\mathbf{d}} \alpha_q$ such that $\gamma_1 + \cdots + \gamma_i < \alpha_i$, $\gamma_1 + \cdots + \gamma_i <_{\mathbf{d}} \alpha_q$ for $1 \leq i < q$ and $\gamma_1 + \cdots + \gamma_q = \alpha_q$ we denote

$$I_{\gamma_1,\ldots,\gamma_q}^{(q)} = \prod_{e=1}^{q}\prod_{t=0}^{s}(\mathbf{n}_e^{[d_t]})^{\gamma_{et}}. \textit{Proposition } 1.7.2 \textit{ implies}: \ I = \sum_{q=1}^{r}\sum_{\gamma_1,\ldots,\gamma_q} I_{\gamma_1,\ldots,\gamma_q}^{(q)}.$$

Let $\mathbf{m} = (x_1, \ldots, x_n) \subset S$ the irrelevant ideal of S. We have:

$$(I :_S \mathbf{m}) = \bigcap_{j=1}^{n}(I : x_j) = \bigcap_{j=1}^{n}((\sum_{q=1}^{r}\sum_{\gamma_1,\ldots,\gamma_q} I_{\gamma_1,\ldots,\gamma_q}^{(q)}) : x_j) = \bigcap_{j=1}^{n}(\sum_{q=1}^{r}\sum_{\gamma_1,\ldots,\gamma_q}(I_{\gamma_1,\ldots,\gamma_q}^{(q)} : x_j)).$$

On the other hand, if $x_j \in \mathbf{n}_p$ for some $1 \leq p \leq q$ then

$$J^{(q),j}_{\gamma_1,\ldots,\gamma_q} := (I^{(q)}_{\gamma_1,\ldots,\gamma_q} : x_j) = \prod_{e\neq p}^{q}\prod_{t=0}^{s}(\mathbf{n}_e^{[d_t]})^{\gamma_{et}}\mathbf{n}_{\mathbf{p},\hat{\mathbf{j}}}{}^{[d_t]}(\mathbf{n}_p^{[d_t]})^{\gamma_{pt}-1}(\sum_{\gamma_{pt}>0}\prod_{j\neq t}(\mathbf{n}_e^{[d_t]})^{\gamma_{jt}}),$$

where $\mathbf{n}_{\mathbf{p},\hat{\mathbf{j}}}{}^{[d_t]} = (x^{d_t}_{i_{p-1}+1}, \ldots, x_j^{d_t-1}, \ldots, x_{i_p}^{d_t})$ and $\mathbf{n}_{\mathbf{p},\hat{\mathbf{j}}}{}^{[d_t]}(\mathbf{n}^{[d_t]})^{\gamma_{pt}-1} := S$ if $\gamma_{pt} = 0$. Thus

$$(I :_S \mathbf{m}) = \left(\sum_{q^1=1}^{r}\sum_{\gamma_1^1,\ldots,\gamma_q^1}\right)\cdots\left(\sum_{q^n=1}^{r}\sum_{\gamma_1^n,\ldots,\gamma_q^n}\right)\bigcap_{j=1}^{n} J^{(q^j),j}_{\gamma_1^j,\ldots,\gamma_q^j},$$

where for a given q^j, we have that $\gamma_1^j, \ldots, \gamma_q^j \leq_{\mathbf{d}} \alpha_q$ are some nonnegative integers such that $\gamma_1^j + \cdots + \gamma_i^j < \alpha_i$, $\gamma_1^j + \cdots + \gamma_i^j <_{\mathbf{d}} \alpha_q$ for $1 \leq i < q^j$ and $\gamma_1^j + \cdots + \gamma_q^j = \alpha_q$.

Proposition 1.7.5. *Let $n \geq 2$ and let $1 \leq i_1 < i_2 < \cdots < i_r = n$ some integers. Let $\alpha_1 < \alpha_2 < \cdots < \alpha_r$ some positive integers. Let $s_q = max\{t|\ \alpha_{qt} > 0\}$ for any $1 \leq q \leq r$. We consider the ideal $I = \sum_{q=1}^{r} I_q$, where $I_q =< x_{i_q}^{\alpha_q} >_{\mathbf{d}}$. Then, we have: $reg(I) \leq reg(I_r)$.*

Proof. Denote $e = max\{reg(I_1), \ldots, reg(I_r)\}$. Since a $\mathbf{d}$-fixed ideal is an ideal of Borel type, by Corollary 1.2.13 it follows that $reg(I) \leq e$. On the other hand, if we denote $s_q = max\{t|\ \alpha_{qt} > 0\}$ for any $1 \leq q \leq r$, from Theorem 1.6.1 we get $reg(I_q) = \alpha_{qs_q}d_{s_q} + (i_q - 1)(d_{s_q} - 1)$, thus $max\{reg(I_1), \ldots, reg(I_r)\} = reg(I_r)$. In conclusion, $reg(I) \leq reg(I_r)$. □

Proposition 1.7.6. *With the above notations, for any $1 \leq q \leq r$ we have:*

$$(I_q : \mathbf{m}_q)+(I_1+\cdots+I_q) \subset ((I_1+\cdots+I_q) : \mathbf{m}_q) \subset ((I_1+\cdots+I_q) : \mathbf{n}_q) = (I_q : \mathbf{n}_q)+(I_1+\cdots+I_q).$$

Proof. Fix $1 \leq q \leq r$. The first two inclusions are obvious. In order to prove the last equality, it is enough to show that $((I_1 + \cdots + I_q) : x_j) \subset (I_q : x_j) + (I_1 + \cdots + I_q)$ for any $x_j \in \mathbf{n}_q$. Indeed, suppose $u \in ((I_1 + \cdots + I_q) : x_j)$, therefore $x_j \cdot u \in I_1 + \cdots + I_q$. If $x_j \cdot u \notin I_q$ it follows that $x_j \cdot u \in I_e$ for some $e < q$. Thus $u \in I_e$, since x_j does not appear in a minimal generators of I_e. □

Chapter 2

Generic initial ideal for complete intersections.

2.1 Main results.

Let $S = K[x_1, \ldots, x_n]$ and let $I = (f_1, \ldots, f_n) \subset S$ be an ideal generated by a regular sequence of homogeneous polynomials. We say that a homogeneous polynomial f of degree d is *semiregular* for S/I if the maps $(S/I)_t \xrightarrow{\cdot f} (S/I)_{t+d}$ are either injective, either surjective for all $t \geq 0$. We say that S/I has the *weak Lefschetz property* (WLP) if there exists a linear form $\ell \in S$, semiregular on S/I, in which case we say that ℓ is a weak Lefschetz element for S/I. We say that S/I has the *strong Lefschetz property* (SLP) if there exists a linear form $\ell \in S$ such that ℓ^b is semiregular on S/I for all integer $b \geq 1$. In this case, we say that ℓ is a strong Lefschetz element for S/I.

We say that a property (P) holds for a *generic* sequence of homogeneous polynomials $f_1, f_2, \ldots, f_n \in S = K[x_1, x_2, \ldots, x_n]$ of given degrees $d_1, d_2, \ldots, d_n$ if there exists a nonempty open Zariski subset $U \subset S_{d_1} \times S_{d_2} \times \cdots \times S_{d_n}$ such that for every $(f_1, f_2, \ldots, f_n) \in U$ the property (P) holds.

Conjecture.(Moreno) If $f_1, f_2, \ldots, f_n \in S = K[x_1, \ldots, x_n]$ is a generic sequence of homogeneous polynomials of given degrees $d_1, d_2, \ldots, d_n$, $I = (f_1, \ldots, f_n)$ and J is the initial ideal of I with respect to the revlex order, then J is an almost revlex ideal, i.e. if $u \in J$ is a minimal generator of J then every monomial of the same degree which preceeds u must be in J as well.

Theorem 2.1.1. *If $f_1, f_2, f_3 \in S = K[x_1, x_2, x_3]$ is a regular sequence of homogeneous polynomials of given degrees d_1, d_2, d_3 and $I = (f_1, f_2, f_3)$ such that S/I has the (SLP) then $J = Gin(I)$ is uniquely determined and is an almost reverse lexicographic ideal.*

Proof. The theorem is a direct consequence of the Propositions 2.2.3, 2.2.8, 2.3.3, 2.3.8, 2.3.13 and 2.3.17. $\square$

Theorem 2.1.2. *The conjecture Moreno is true for $n = 3$ (and $char(K) = 0$).*

Proof. Notice that (SLP) is an open condition. Also, the condition that a sequence of homogeneous polynomial is regular is an open condition. It follows, using Theorem 2.1.1, that for a generic sequence f_1, f_2, f_3 of homogeneous polynomials of given degrees d_1, d_2, d_3, $J = Gin(I)$ is almost revlex, where $I = (f_1, f_2, f_3)$. But the definition of the generic initial ideal implies to choose a generic change of variables, and therefore for a generic sequence f_1, f_2, f_3 of homogeneous polynomials of given degrees $in(I)$ is almost revlex, as required. □

From now on, we fix some integers $2 \leq d_1 \leq d_2 \leq d_3$. Let $f_1, f_2, f_3 \in S = K[x_1, x_2, x_3]$ be a regular sequence of homogeneous polynomials of given degrees d_1, d_2, d_3 and $I = (f_1, f_2, f_3)$ such that S/I has the (SLP) and let $J = Gin(I)$ be the generic initial ideal of I, with respect to revers lexicographic order. We will use these conventions in the sections 2 and 3 of this chapter.

Remark 2.1.3. *In order to compute J we will use the fact that J is a strongly stable ideal, i.e. for any monomial $u \in J$ and any indices $j < i$, if $x_i | u$ then $x_j u/x_i \in J$. Also, a theorem of Wiebe (see [34]) states that S/I has (SLP) if and only if x_3 is a strong Lefschetz element for S/J. We need to consider several cases: I. $d_1 + d_2 \leq d_3 + 1$ with 2 subcases $d_1 = d_2 < d_3$, $d_1 < d_2 < d_3$ (section 2) and II. $d_1 + d_2 > d_3 + 1$ with 4 subcases: $d_1 = d_2 = d_3$, $d_1 = d_2 < d_3$, $d_1 < d_2 = d_3$, $d_1 < d_2 < d_3$ (section 3).*

The construction of J in all cases, follows the next procedure. For any nonnegative integer k, we denote by J_k the set of monomials of degree k in J. We can easily compute the cardinality of each J_k from the Hilbert series of S/J. We denote $Shad(J_k) = \{x_i u : 1 \leq i \leq n,\ u \in J_k\}$. We begin with $J_0 = \emptyset$ and we pass from J_k to J_{k+1} noticing that $J_{k+1} = Shad(J_k) \cup$ eventually some new monomial(s) (exactly $|J_{k+1}| - |Shad(J_k)|$ new monomials). The fact that J is strongly stable and that x_3 is a strong Lefschetz element for S/J tell us what we need to add to $Shad(J_k)$ in order to obtain J_{k+1}. We continue this procedure until $k = d_1 + d_2 + d_3 - 2$ since $J_k = S_k$(=the set of all monomials of degree k) for any $k \geq d_1 + d_2 + d_3 - 2$ and so we cannot add any new monomials in larger degrees. J is the ideal generated by all monomials added to $Shad(J_k)$ at some step k. We will present detailed this construction only in the subcase $d_1 = d_2 < d_3$ of the case I.$d_1 + d_2 \leq d_3 + 1$ (Proposition 2.2.3), the other cases being presented in sketch, but the reader can easily complete the proofs. The condition that S/I has (SLP) is needed. Indeed, there are monomial ideals J such that S/J and S/I have the same Hilbert series, J is strongly stable and x_3 is a weak Lefschetz element for S/J, but is not strong Lefschetz. For example,

$$J = (x_1\{x_1, x_2\}^2,\ x_2^4,\ x_1^2x_3^2,\ x_2^3, x_3^2,\ x_1x_2x_3^3,\ x_1x_3^4,\ x_2^2x_3^4,\ x_2x_3^5,\ x_3^7)$$

has $H(S/J, t) = (1+t+t^2)^3$, i.e. the Hilbert series of a complete intersection $S/(f_1, f_2, f_3)$, with f_1, f_2, f_3 homogeneous of degree 3. Also, J is strongly stable and x_3 is a weak Lefschetz element for S/J, but not strong Lefschetz, since the map $(S/J)_2 \xrightarrow{\cdot x_3^2} (S/J)_4$ is not injective and $|(S/J)_2| = |(S/J)_4| = 6$.

2.2 Case $d_1 + d_2 \leq d_3 + 1$.

- Subcase $d_1 = d_2 < d_3$.

Proposition 2.2.1. Let $2 \leq d := d_1 = d_2 < d_3$ be positive integers such that $2d \leq d_3 + 1$. The Hilbert function of the standard graded complete intersection $A = K[x_1, x_2, x_3]/I$, where I is the ideal generated by f_1, f_2, f_3, with f_i homogeneous polynomials of degree d_i, for all i, with $1 \leq i \leq 3$, has the form:

1. $H(A, k) = \binom{k+2}{k}$, for $k \leq d - 1$.
2. $H(A, k) = \binom{d+1}{2} + \sum_{i=1}^{j}(d - i)$, for $k = d - 1 + j$, where $0 \leq j \leq d - 1$.
3. $H(A, k) = d^2$, for $2d - 2 \leq k \leq d_3 - 1$.
4. $H(A, k) = H(A, 2d + d_3 - 3 - k)$ for $k \geq d_3$.

Proof. It follows from [31, Lemma 2.9(a)]. □

Corollary 2.2.2. In the conditions of Proposition 2.2.1, let $J = Gin(I)$ be the generic initial ideal of I with respect to the reverse lexicographic order. If we denote by J_k the set of monomials of J of degree k, then:

1. $J_k = \emptyset$, for $k \leq d - 1$.
2. $|J_k| = j(j + 1)$, for $k = d - 1 + j$, where $0 \leq j \leq d - 1$.
3. $|J_k| = d(d - 1) + 2dj + \frac{j(j-1)}{2}$, for $k = 2d - 2 + j$, where $0 \leq j \leq d_3 + 1 - 2d$.
4. $|J_k| = \frac{d_3(d_3+1)}{2} - d^2 + jd_3 + j(j + 1)$, for $k = d_3 - 1 + j$, where $0 \leq j \leq d - 1$.
5. $|J_k| = \frac{d_3(d_3-1)}{2} + d(d_3 - 1) + j(d_3 + 2d)$, for $k = d + d_3 - 2 + j$, where $0 \leq j \leq d$.
6. $J_k = S_k$, for $k \geq 2d + d_3 - 2$, where S_k is the set of monomials of degree k.

Proof. Using that $|J_k| = |S_k| - H(S/J, k)$, together with the general fact that $H(S/J, k) = H(S/I, k)$, the proof follows immediately from Proposition 2.2.1. □

I would like to express my gratitude to Dr.Marius Vladoiu for his help on the proof of the following proposition and, in general, for his help on the sections 2.2 and 2.3.

Proposition 2.2.3. Let $2 \leq d := d_1 = d_2 < d_3$ be positive integers such that $2d \leq d_3 + 1$. Let $f_1, f_2, f_3 \in K[x_1, x_2, x_3]$ be a regular sequence of homogeneous polynomials of degrees d_1, d_2, d_3. If $I = (f_1, f_2, f_3)$, and $J = Gin(I)$, the generic initial ideal with respect to the reverse lexicographic order, and S/I has (SLP), then:

$$J = (x_1^d, x_1^{d-j-1}x_2^{2j+1} \; for \; 0 \leq j \leq d-1, x_2^{2d-2j-2}x_3^{d_3-2d+2j+2}\{x_1, x_2\}^j \; for \; 0 \leq j \leq d-2,$$
$$x_3^{d_3+2j-2}\{x_1, x_2\}^{d-j} \; for \; 1 \leq j \leq d).$$

Proof. We have $|J_d| = 2$, hence $J_d = \{x_1^{d-1}\{x_1, x_2\}\}$, since J is a strongly stable ideal. Therefore:

$$Shad(J_d) = \{x_1^{d-1}\{x_1, x_2\}^2, x_1^{d-1}x_3\{x_1, x_2\}\},$$

Now we have two possibilities to analyze: $d = 2$ and $d \geq 3$. First, suppose $d \geq 3$.

Using the formulas from Corollary 2.2.2 we have $|J_{d+1}| - |Shad(J_d)| = 1$, so there is only one generator to add to the set $Shad(J_d)$ in order to obtain J_{d+1}. Since J is strongly stable, we have only two possibilities: $x_1^{d-2}x_2^3$ or $x_1^{d-1}x_3^2$. We cannot have $x_1^{d-1}x_3^2$, since otherwise the application

$$(S/J)_{d-1} \xrightarrow{\cdot x_3^2} (S/J)_{d+1},$$

with $|(S/J)_{d-1}| < |(S/J)_{d+1}|$ (see Proposition 2.2.1) would not be injective ($0 \neq x_1^{d-1} \in (S/J)_{d-1}$ and is mapped to 0), which is a contradiction to the fact that x_3 is a strong Lefschetz element for S/J. Hence:

$$J_{d+1} = \{x_1^{d-2}\{x_1, x_2\}^3, x_1^{d-1}x_3\{x_1, x_2\}\}.$$

We prove by induction on j, with $1 \leq j \leq d-2$, that:

$$J_{d+j} = Shad(J_{d+j-1}) \cup \{x_1^{d-j-1}x_2^{2j+1}\} =$$

$$= \{x_1^{d-j-1}\{x_1, x_2\}^{2j+1}, x_1^{d-j}x_3\{x_1, x_2\}^{2j-1}, \ldots, x_1^{d-1}x_3^j\{x_1, x_2\}\}.$$

The assertion was checked above for $j = 1$. Assume now that the statement is true for some $j < d-2$. Then $Shad(J_{d+j})$ is the following set:

$$\{x_1^{d-j-1}\{x_1, x_2\}^{2j+2}, x_1^{d-j-1}x_3\{x_1, x_2\}^{2j+1}, x_1^{d-j}x_3^2\{x_1, x_2\}^{2j-1}, \ldots, x_1^{d-1}x_3^{j+1}\{x_1, x_2\}\}.$$

We have $|J_{d+j+1}| - |Shad(J_{d+j})| = 1$, so we must add only one generator to $Shad(J_{d+j})$ to get J_{d+j+1}. The ideal J, being strongly stable, allows only two possibilities, namely $x_1^{d-j-2}x_2^{2j+3}$ or $x_1^{d-2}x_2^2x_3^{j+1}$. The second one is not allowed because the application

$$(S/J)_d \xrightarrow{\cdot x_3^{j+1}} (S/J)_{d+j+1}$$

would not be injective ($0 \neq x_1^{d-2}x_2^2 \in (S/J)_d$ and is mapped to 0), x_3 being a strong-Lefschetz element for S/J. Therefore, we must add $x_1^{d-j-2}x_2^{2j+3}$ and our claim is proved. In particular, we obtain

$$J_{2d-2} = \{x_1\{x_1, x_2\}^{2d-3}, x_1^2x_3\{x_1, x_2\}^{2d-5}, \ldots, x_1^{d-1}x_3^{d-2}\{x_1, x_2\}\}.$$

In order to compute J_k, with $2d-2 \leq k \leq d_3$, we must consider two posibilities.

- 1. $d_3 = 2d - 1$.

Since $|J_{2d-1}| - |Shad(J_{2d-2})| = 2$, there are two generators to add to $Shad(J_{2d-2})$. We prove that these generators are $x_2^{2d-1}, x_2^{2d-2}x_3$. Assuming by contradiction that we have other generators, since J is strongly stable, it follows that there is at least one generator from the set $\{x_1x_2^{2d-4}x_3^2, x_1^2x_2^{2d-6}x_3^3, \ldots, x_1^{d-2}x_2^2x_3^{d-1}\}$. Then, the application $(S/J)_{2d-3} \xrightarrow{\cdot x_3^2} (S/J)_{2d-1}$ would not be injective, a contradiction since x_3 is a strong Lefschetz element for S/J. Therefore

$$J_{2d-1} = J_{d_3} = \{\{x_1, x_2\}^{2d-1}, x_3\{x_1, x_2\}^{2d-2}, \ldots, x_1^{d-1}x_3^{d-1}\{x_1, x_2\}\}.$$

- 2. $d_3 > 2d - 1$.

Since $|J_{2d-1}| - |Shad(J_{2d-2})| = 1$, there is only one generator to add to $Shad(J_{2d-2})$, which can be selected from the set $\{x_2^{2d-1}, x_1x_2^{2d-4}x_3^2, x_1^2x_2^{2d-6}x_3^3, \ldots, x_1^{d-2}x_2^2x_3^{d-1}\}$ because J is strongly stable. In a similar manner to what we have done above can be shown that, x_3 being a strong Lefschetz element, leaves us as unique possibility x_2^{2d-1}, therefore:

$$J_{2d-1} = \{\{x_1, x_2\}^{2d-1}, x_1x_3\{x_1, x_2\}^{2d-3}, \ldots, x_1^{d-1}x_3^{d-1}\{x_1, x_2\}\}.$$

One can easily show, using induction on $1 \leq j \leq d_3 - 2d$, if case, that $|J_{2d-1+j}| = |Shad(J_{2d-2+j})|$ and J_{2d-1+j} is the set

$$\{\{x_1, x_2\}^{2d-1+j}, \ldots, x_3^j\{x_1, x_2\}^{2d-1}, x_3^{j+1}x_1\{x_1, x_2\}^{2d-2}, \ldots, x_3^{d+j-1}x_1^{d-1}\{x_1, x_2\}\}.$$

In particular, we obtain that J_{d_3-1} is the set

$$\{\{x_1, x_2\}^{d_3-1}, \ldots, x_3^{d_3-2d}\{x_1, x_2\}^{2d-1}, x_3^{d_3-2d+1}x_1\{x_1, x_2\}^{2d-3}, \ldots, x_3^{d_3-d-1}x_1^{d-1}\{x_1, x_2\}\}.$$

Since $|J_{d_3}| - |Shad(J_{d_3-1})| = 1$, the generator which has to be add to $Shad(J_{d_3-1})$ can be selected from the set $\{x_2^{2d-2}x_3^{d_3-2d+2}, x_1x_2^{2d-4}x_3^{d_3-2d+3}, \ldots, x_1^{d-2}x_2^2x_3^{d_3-d}\}$ such that J is strongly stable. The generator is $x_2^{2d-2}x_3^{d_3-2d+2}$, otherwise the application $(S/J)_{2d-3} \xrightarrow{\cdot x_3^{d_3-2d+3}} (S/J)_{d_3}$ is not injective, a contradiction, since x_3 is a strong Lefschetz element for S/J. Hence, we get that J_{d_3} is

$$\{\{x_1, x_2\}^{d_3}, \ldots, x_3^{d_3-2d+2}\{x_1, x_2\}^{2d-2}, x_1^2x_3^{d_3-2d+3}\{x_1, x_2\}^{2d-5}, \ldots, x_3^{d_3-d}x_1^{d-1}\{x_1, x_2\}\},$$

and one can check that is the same formula as in 1.($d_3 = 2d - 1$).

Now, we show by induction on $1 \leq j \leq d - 2$ that

$$J_{d_3+j} = Shad(J_{d_3-1+j}) \cup \{x_2^{2d-2j-2}x_3^{d_3-2d+2j+2}\{x_1, x_2\}^j\}.$$

Indeed, for $j = 1$, $|J_{d_3+1}| - |Shad(J_{d_3})| = 2$ and the generators which must be added are $x_1x_2^{2d-4}x_3^{d_3-2d+4}, x_2^{2d-3}x_3^{d_3-2d+4}$. If not, since J is strongly stable, then at least one of the generators belongs to the set $\{x_1^2x_2^{2d-6}x_3^{d_3-2d+5}, \ldots, x_1^{d-2}x_2^2x_3^{d_3-d+1}\}$ (for $d = 3$ this is the emptyset). but then the map $(S/J)_{2d-4} \xrightarrow{x_3^{d_3-2d+5}} (S/J)_{d_3+1}$ is not injective, contradiction.

Assume now that we proved the assertion for some $j < d-2$. Then $|J_{d_3+j+1}| - |Shad(J_{d_3+j})| = j+2$ and the new generators are $x_2^{2d-2j-4}x_3^{d_3-2d+2j+4}\{x_1,x_2\}^{j+1}$. Indeed, if not, since J is strongly stable, then at least one of the generators belongs to the set $\{x_1^{j+2}x_2^{2d-2j-6}x_3^{d_3-2d+2j+5},\ldots,x_1^{d-2}x_2^2x_3^{d_3-d+j+1}\}$ (for $d=3$ this is the emptyset...) but then the map $(S/J)_{2d-j-4} \xrightarrow{x_3^{d_3-2d+2j+5}} (S/J)_{d_3+j+1}$ is not injective, contradiction, and we are done. Hence,

$$J_{d_3+d-2} = \{\{x_1,x_2\}^{d_3+d-2}, x_3\{x_1,x_2\}^{d+d_3-3},\ldots,x_3^{d_3-2}\{x_1,x_2\}^d\}.$$

We prove by induction on $1 \leq j \leq d$ that $J_{d+d_3-2+j} = Shad(J_{d+d_3-3+j}) \cup x_3^{d_3+2j-2}\{x_1,x_2\}^{d-j}$. If $j=1$ then $|J_{d+d_3-1}| - |Shad(J_{d+d_3-2})| = d$ so we must add d generators, which are precisely the elements of the set $x_3^{d_3}\{x_1,x_2\}^{d-1}$. Indeed, if we have a generator which does not belong to the set it is divisible by $x_3^{d_3+1}$ and therefore the map $(S/J)_{d-2} \xrightarrow{\cdot x_3^{d_3+1}} (S/J)_{d_3+d-1}$ is not injective, which is a contradiction with x_3 is a strong Lefschetz element for S/J (the map has to be bijective). The induction step is similar and finally we obtain that $J_{d_3+2d-2} = S_{d_3+2d-2}$ and thus we cannot add new minimal generators of J in degree $> d_3+2d-2$.

In order to complete the proof we must consider now $d=2$. The hypothesis implies $d_3 \geq 3$. We already seen that $J_2 = \{x_1^2, x_1x_2\}$ and $Shad(J_2) = \{x_1\{x_1,x_2\}^2, x_1x_3\{x_1,x_2\}\}$.

Using the formulas from Corollary 2.2.2 we have $|J_3| - |Shad(J_2)| = 1$, so there is only one generator to add to the set $Shad(J_d)$ in order to obtain J_{d+1}.

Since J is strongly stable, we have only two possibilities: x_2^3 or $x_1x_3^2$. We can not have $x_1x_3^2$, since otherwise the application

$$(S/J)_1 \xrightarrow{\cdot x_3^2} (S/J)_3,$$

with $|(S/J)_1| < |(S/J)_3|$ (see Proposition 2.2.1) would not be injective ($0 \neq x_1 \in (S/J)_1$ and is mapped to 0), which is a contradiction to the fact that x_3 is a strong Lefschetz element for S/J. Hence:

$$J_3 = \{\{x_1,x_2\}^3, x_1x_3\{x_1,x_2\}\}.$$

Assume now $d_3 \geq 4$. One can easily show, using induction on $1 \leq j \leq d_3-4$, if case, that $|J_{3+j}| = |Shad(J_{2+j})|$ and J_{3+j} is the set

$$\{\{x_1,x_2\}^{3+j},\ldots,x_3^j\{x_1,x_2\}^3, x_3^{j+1}x_1\{x_1,x_2\}\}.$$

In particular, we obtain that

$$J_{d_3-1} = \{\{x_1,x_2\}^{d_3-1},\ldots,x_3^{d_3-4}\{x_1,x_2\}^3, x_3^{d_3-3}x_1\{x_1,x_2\}\}.$$

Since $|J_{d_3}| - |Shad(J_{d_3-1})| = 1$, the generator which has to be add to $Shad(J_{d_3-1})$ is exactly $x_2^2x_3^{d_3-2}$ such that J is strongly stable. Hence, we get

$$J_{d_3} = \{\{x_1,x_2\}^{d_3},\ldots,x_3^{d_3-3}\{x_1,x_2\}^3, x_1x_3^{d_3-2}\{x_1,x_2\}\},$$

and one can check that is the same formula as in the case $d_3 = 3$.

Since $|J_{d_3+1}| - |Shad(J_{d_3})| = 1$, there is only one generator to add to the set $Shad(J_{d_e})$ in order to obtain J_{d_3+1}. Since J is strongly stable, we have only two possibilities: $x_2^2 x_3^{d_3-1}$ or $x_1 x_3^{d_3}$. We can not have $x_1 x_3^{d_3}$, since otherwise the application

$$(S/J)_1 \xrightarrow{\cdot x_3^{d_3}} (S/J)_{d_3+1},$$

with $|(S/J)_1| = |(S/J)_3|$ (see Proposition 2.2.1) would not be injective ($0 \neq x_1 \in (S/J)_1$ and is mapped to 0), which is a contradiction to the fact that x_3 is a strong Lefschetz element for S/J. Hence:

$$J_{d_3+1} = \{\{x_1, x_2\}^{d_3+1}, \ldots, x_3^{d_3-1}\{x_1, x_2\}^2\}.$$

Since $|J_{d_3+2}| - |Shad(J_{d_3+1})| = 2$ and J is strongly stable, we must add $x_1 x_3^{d_3+1}$ and $x_2 x_3^{d_3+1}$ at $Shad(J_{d_3+1})$ in order to obtain J_{d_3+2}. Hence $J_{d_3+2} = S_{d_3+2} \setminus \{x_3^{d_3+2}\}$. Finally, since $J_{d_3+3} = S_{d_3+3}$ we add $x_3^{d_3+3}$ at $Shad(J_{d_3+2})$ and thus we cannot add new minimal generators of J in degree $> d_3 + 2$. □

Corollary 2.2.4. *In the conditions of the above proposition, the number of minimal generators of J is $d^2 + d + 1$.*

Example 2.2.5. *Let $d_1 = d_2 = 3$ and $d_3 = 9$. Proposition 2.2.3 implies:*

$$J = (x_1^3,\ x_1^2 x_2,\ x_1 x_2^3,\ x_2^5,\ x_2^4 x_3^5,\ x_1 x_2^2 x_3^7,\ x_2^3 x_3^7,\ x_3^9\{x_1, x_2\}^2,\ x_3^{11}\{x_1, x_2\},\ x_3^{13}).$$

- Subcase $d_1 < d_2 < d_3$.

Proposition 2.2.6. Let $2 \leq d_1 < d_2 < d_3$ be positive integers such that $d_1 + d_2 \leq d_3 + 1$. The Hilbert function of the standard graded complete intersection $A = K[x_1, x_2, x_3]/I$, where I is the ideal generated by f_1, f_2, f_3, with f_i homogeneous polynomials of degree d_i, for all i, with $1 \leq i \leq 3$, has the form:

1. $H(A, k) = \binom{k+2}{k}$, for $k \leq d_1 - 1$.
2. $H(A, k) = \binom{d_1+1}{2} + jd_1$, for $k = j + d_1 - 1$, where $0 \leq j \leq d_2 - d_1$.
3. $H(A, k) = \binom{d_1+1}{2} + d_1(d_2 - d_1) + \sum_{i=1}^{j}(d_1 - i)$, for $k = j + d_2 - 1$, where $0 \leq j \leq d_1 - 1$.
4. $H(A, k) = d_1 d_2$, for $d_1 + d_2 - 2 \leq k \leq d_3 - 1$.
5. $H(A, k) = H(A, d_1 + d_2 + d_3 - 3 - k)$ for $k \geq d_3$.

Proof. It follows from [31, Lemma 2.9(a)]. □

Corollary 2.2.7. In the conditions of Proposition 2.2.6, let $J = Gin(I)$ be the generic initial ideal of I with respect to the reverse lexicographic order. If we denote by J_k the set of monomials of J of degree k, then:

1. $|J_k| = 0$, for $k \leq d - 1$.
2. $|J_k| = j(j+1)/2$, for $k = j + d_1 - 1$, where $0 \leq j \leq d_2 - d_1$.
3. $|J_k| = \frac{(d_2-d_1)((d_2-d_1-1))}{2} + j(d_2 - d_1) + j(j+1)$, for $k = j + d_2 - 1$, where $0 \leq j \leq d_1 - 1$.
4. $|J_k| = \frac{d_1^2+d_2^2-d_1-d_2}{2} + j(d_1 + d_2) + \frac{j(j-1)}{2}$, for $k = j + d_1 + d_2 - 2$, where $0 \leq j \leq d_3 - d_1 - d_2 + 1$.
5. $|J_k| = \frac{d_3^2+d_3-2d_1d_2}{2} + jd_3 + j(j+1)$, for $k = j + d_3 - 1$ where $0 \leq j \leq d_1 - 1$.
6. $|J_k| = \frac{(d_1+d_3)(d_1+d_3-1)+d_1^2-d_1-2d_1d_2}{2} + j(d_3 + 2d_1) + \frac{j(j-1)}{2}$, for $k = j + d_1 + d_3 - 2$, where $0 \leq j \leq d_2 - d_1$.
7. $|J_k| = \frac{(d_2+d_3)(d_2+d_3-1)+d_1(d_1-1)}{2} + j(d_1+d_2+d_3)$, for $k = d_2+d_3-2$, where $0 \leq j \leq d_1-1$.
8. $J_k = S_k$, for $k \geq 3d - 2$.

Proposition 2.2.8. Let $2 \leq d_1 < d_2 < d_3$ be positive integers such that $d_1 + d_2 \leq d_3 + 1$. Let $f_1, f_2, f_3 \in K[x_1, x_2, x_3]$ be a regular sequence of homogeneous polynomials of degrees d_1, d_2, d_3. If $I = (f_1, f_2, f_3)$,$J = Gin(I)$, the generic initial ideal with respect to the reverse lexicographic order, and S/I has (SLP),then:

$$J = (x_1^{d_1}, x_1^{d_1-j}x_2^{d_2-d_1+2j-1}\ for\ 1 \leq j \leq d_1 - 1,\ x_2^{d_1+d_2-1},\ x_2^{d_1+d_2-2}x_3^{d_3-d_1-d_2+2},$$
$$x_3^{d_3-d_1-d_2+2j+2}x_2^{d_1+d_2-2j-2}\{x_1, x_2\}^j\ for\ 1 \leq j \leq d_1 - 2,$$
$$x_3^{d_3+d_1-d_2-2+2j}x_2^{d_2-d_1+1-j}\{x_1, x_2\}^{d_1-1}\ for\ 1 \leq j \leq d_2 - d_1,$$
$$x_3^{d_3+d_2-d_1+2j-2}\{x_1, x_2\}^{d_1-j}\ for\ 1 \leq j \leq d_1).$$

Proof. We have $|J_{d_1}| = 1$, hence $J_{d_1} = \{x_1^{d_1}\}$, since J is a strongly stable ideal. Therefore:

$$Shad(J_{d_1}) = \{x_1^{d_1}\{x_1, x_2\}, x_1^{d_1}x_3\}.$$

Assume $d_2 > d_1 + 1$. Since $|J_{d_1+1}| = |Shad(J_{d_1})|$ from the formulas of 2.2.7, if follows $J_{d_1+1} = Shad(J_{d_1})$. We prove by induction on $1 \leq j \leq d_2 - d_1 - 1$ that

$$J_{d_1+j} = Shad(J_{d_1+j-1}) = \{x_1^{d_1}\{x_1, x_2\}^j, x_3x_1^{d_1}\{x_1, x_2\}^{j-1}, \dots, x_3^jx_1^{d_1}\}.$$

Indeed, the case $j = 1$ is already proved. Suppose the assertion is true for some $j < d_2 - d_1 - 1$. Since $|J_{d_1+j+1}| - |Shad(J_{d_1+j})| = 0$ it follows that

$$J_{d_1+j+1} = Shad(J_{d_1+j}) = \{x_1^{d_1}\{x_1, x_2\}^{j+1}, x_3x_1^{d_1}\{x_1, x_2\}^j, \dots, x_3^{j+1}x_1^{d_1}\}$$

thus we are done. In particular, we get

$$J_{d_2-1} = \{x_1^{d_1}\{x_1, x_2\}^{d_2-d_1-1}, x_3x_1^{d_1}\{x_1, x_2\}^{d_2-d_1-2}, \ldots, x_3^{d_2-d_1-1}x_1^{d_1}\}$$

which is the same formula as in the case $d_2 = d_1 + 1$.

We have $|J_{d_2}| - |Shad(J_{d_2-1})| = 1$ so we must add a new generator to $Shad(J_{d_2-1})$ to obtain J_{d_2}. Since J is strongly stable and x_3 is a strong Lefschetz element for S/J this new generator is $x_1^{d_1-1}x_2^{d_2-d_1+1}$, therefore

$$J_{d_2} = \{x_1^{d_1-1}\{x_1, x_2\}^{d_2-d_1+1}, x_3x_1^{d_1}\{x_1, x_2\}^{d_2-d_1-1}, \ldots, x_3^{d_2-d_1}x_1^{d_1}\}.$$

Assume $d_1 > 2$. We prove by induction on $1 \leq j \leq d_1 - 1$ that

$$J_{d_2-1+j} = Shad(J_{d_2-2+j}) \cup \{x_1^{d_1-j}x_2^{d_2-d_1+2j-1}\} = \{x_1^{d_1-j}\{x_1, x_2\}^{d_2-d_1+2j-1},$$

$$x_3x_1^{d_1-j+1}\{x_1, x_2\}^{d_2-d_1+2j-3}, \ldots, x_3^jx_1^{d_1}\{x_1, x_2\}^{d_2-d_1-1}, \ldots, x_3^{d_2-d_1+j-1}x_1^{d_1}\}.$$

The assertion was proved for $j = 1$. Suppose $1 \leq j < d_1 - 1$ and the assertion is true for j. We have $|J_{d_2+j}| - |Shad(J_{d_2-1+j})| = 1$, thus we must add a new generator to $Shad(J_{d_2-1+j})$ in order to obtain J_{d_2+j} and since J is strongly stable and x_3 is a strong Lefschetz element for S/J, this is $x_1^{d_1-j-1}x_2^{d_2-d_1+2j+1}$ and we are done. In particular, we obtain:

$$J_{d_1+d_2-2} = \{x_1\{x_1, x_2\}^{d_1+d_2-3}, x_3x_1^2\{x_1, x_2\}^{d_1+d_2-5}, \ldots,$$

$$x_3^{d_1-1}x_1^{d_1}\{x_1, x_2\}^{d_2-d_1-1}, x_3^{d_1}x_1^{d_1}\{x_1, x_2\}^{d_2-d_1-2}, \ldots, x_3^{d_2-2}x_1^{d_1}\}$$

and one can check that is the same expression as in the case $d_1 = 2$.

In order to compute J_k, with $2d - 2 \leq k \leq d_3$, we must consider two possibilities.

- 1.$d_3 = d_1 + d_2 - 1$.

Since $|J_{d_1+d_2-1}| - |Shad(J_{d_1+d_2-2})| = 2$, there are two generators to add to $Shad(J_{d_1+d_2-2})$ to get $J_{d_1+d_2-1}$, but on the other hand J is strongly stable and x_3 is a strong Lefschetz element for S/J so these generators must be $x_2^{d_1+d_2-1}, x_2^{d_1+d_2-2}x_3$. Therefore:

$$J_{d_1+d_2-1} = J_{d_3} = \{\{x_1, x_2\}^{d_1+d_2-1}, x_3\{x_1, x_2\}^{d_1+d_2-2}, \ldots, x_1^{d_1}x_3^{d_2-1}\}.$$

- 2.$d_3 > d_1 + d_2 - 1$.

Since $|J_{d_1+d_2-1}| - |Shad(J_{d_1+d_2-2})| = 1$, there is only one generator to add to $Shad(J_{2d-2})$, which is precisely $x_2^{d_1+d_2-1}$ since J is strongly stable and x_3 is a strong Lefschetz element for S/J. Therefore

$$J_{d_1+d_2-1} = \{\{x_1, x_2\}^{d_1+d_2-1}, x_1x_3\{x_1, x_2\}^{d_1+d_2-3}, \ldots, x_1^{d_1}x_3^{d_2-1}\}.$$

One can easily show, using induction on $1 \leq j \leq d_3 - d_1 - d_2$, if case, that $|J_{d_1+d_2-1+j}| = |Shad(J_{d_1+d_2-2+j})|$ and $J_{d_1+d_2-1+j}$ is the set

$$\{\{x_1, x_2\}^{j+d_1+d_2-1}, x_3\{x_1, x_2\}^{j+d_1+d_2-2}, \ldots, x_3^j\{x_1, x_2\}^{d_1+d_2-1},$$

$$x_3^{j+1}x_1\{x_1,x_2\}^{d_1+d_2-3},\dots,x_3^{d_1+j}x_1^{d_1}\{x_1,x_2\}^{d_2-d_1-1},\dots,x_3^{j+d_2-1}x_1^{d_1}\}$$

$$So\ J_{d_3-1}=\{\{x_1,x_2\}^{d_3-1},x_3\{x_1,x_2\}^{d_3-2},\dots,x_3^{d_3-d_1-d_2}\{x_1,x_2\}^{d_1+d_2-1},$$
$$x_3^{d_3-d_1-d_2+1}x_1\{x_1,x_2\}^{d_1+d_2-3},\dots,x_3^{d_3-d_2}x_1^{d_1}\{x_1,x_2\}^{d_2-d_1-1},\dots,x_3^{d_3-d_1-1}x_1^{d_1}\}$$

Since $|J_{d_3}|-|Shad(J_{d_3-1})|=1$, J is strongly stable and x_3 is a strong Lefschetz element for S/J, the generator which has to be added to $Shad(J_{d_3-1})$ is $x_2^{d_1+d_2-2}x_3^{d_3-d_1-d_2+2}$. Hence, we get

$$J_{d_3}=\{\{x_1,x_2\}^{d_3},x_3\{x_1,x_2\}^{d_3-1},\dots,x_3^{d_3-d_1-d_2+2}\{x_1,x_2\}^{d_1+d_2-2},$$
$$x_3^{d_3-d_1-d_2+3}x_1^2\{x_1,x_2\}^{d_1+d_2-5},\dots,x_3^{d_3-d_2+1}x_1^{d_1}\{x_1,x_2\}^{d_2-d_1-1},\dots,x_3^{d_3-d_1}x_1^{d_1},\}$$

and one can check that is the same formula as in 1.($d_3=d_1+d_2-1$).

Assume now $d_1>2$. We show by induction on $1\le j\le d_1-2$ that

$$J_{d_3+j}=Shad(J_{d_3-1+j})\cup\{x_1^jx_3^{d_3-d_1-d_2+2j+2}x_2^{d_1+d_2-2j-2},\dots,x_3^{d_3-d_1-d_2+2j+2}x_2^{d_1+d_2-2-j}\}.$$

Indeed, for $j=1$, $|J_{d_3+1}|-|Shad(J_{d_3})|=2$ so we must add two generators to $Shad(J_{d_3})$ in order to obtain J_{d_3+1}. Since J is strongly stable and x_3 is a strong Lefschetz element for S/J, these new generators are $x_1x_3^{d_3-d_1-d_2+4}x_2^{d_1+d_2-2j-4}$ and $x_3^{d_3-d_1-d_2+4}x_2^{d_1+d_2-2j-3}$. Assume now that we proved the assertion for some $j<d_1-2$. Then $|J_{d_3+j+1}|-|Shad(J_{d_3+j})|=j+2$ and since J is strongly stable and x_3 is a strong Lefschetz element for S/J, the new generators are $x_3^{d_3-d_1-d_2+2j+4}x_2^{d_1+d_2-2j-4}\{x_1,x_2\}^{j+1}$ as required. Hence, we get

$$J_{d_1+d_3-2}=\{\{x_1,x_2\}^{d_1+d_3-2},\dots,x_3^{d_3+d_1-d_2-2}\{x_1,x_2\}^{d_2},$$
$$x_1^{d_1}x_3^{d_3+d_1-d_2-1}\{x_1,x_2\}^{d_2},\dots,x_1^{d_1}x_3^{d_3-2}\},$$

and one can check that is the same formula as in the case $d_1=2$.

We have $|J_{d_1+d_3-1}|-|Shad(J_{d_1+d_3-2})|=d_1$ so we must add d_1 new generators to $Shad(J_{d_1+d_3-2})$ and since J is strongly stable and x_3 is a strong Lefschetz element for S/J, they are $x_3^{d_3+d_1-d_2}x_2^{d_2-d_1}\{x_1,x_2\}^{d_1-1}$. Therefore $J_{d_1+d_3-1}$ is the set

$$\{\{x_1,x_2\}^{d_1+d_3-1},..,x_3^{d_3+d_1-d_2}\{x_1,x_2\}^{d_2-1},x_1^{d_1}x_3^{d_3+d_1-d_2+1}\{x_1,x_2\}^{d_2-d_1-2},..,x_1^{d_1}x_3^{d_3-1}\}.$$

Suppose $d_1>2$. We prove by induction on $1\le j\le d_2-d_1$ that:

$$J_{d_1+d_3-2+j}=Shad(J_{d_1+d_3-3+j})\cup\{x_3^{d_3+d_1-d_2-2+2j}x_2^{d_2-d_1+1-j}\{x_1,x_2\}^{d_1-1}\}=$$
$$=\{\{x_1,x_2\}^{d_1+d_3-2+j},\dots,x_3^{d_3+d_1-d_2-2+2j}\{x_1,x_2\}^{d_2-j},$$
$$x_1^{d_1}x_3^{d_3+d_1-d_2-1+2j}\{x_1,x_2\}^{d_2-d_1-j-1},\dots,x_1^{d_1}x_3^{d_3-2+j}\}.$$

We already proved this for $j=1$. Suppose the assertion is true for some $j<d_2-d_1$. Since $|J_{d_1+d_3-1+j}|-|Shad(J_{d_1+d_3-2+j})|=d_1$ we must add d_1 new generators to $Shad(J_{d_1+d_3-2+j})$

and these new generators are $x_3^{d_3+d_1-d_2-2+2j}x_2^{d_2-d_1+1-j}\{x_1,x_2\}^{d_1-1}$ because J is strongly stable and x_3 is a strong Lefschetz element for S/J. In particular, we get:

$$J_{d_2+d_3-2} = \{\{x_1,x_2\}^{d_2+d_3-2}, x_3\{x_1,x_2\}^{d_2+d_3-3}, \ldots, x_3^{d_3+d_2-d_1-2}\{x_1,x_2\}^{d_1}\},$$

which is the same formula as in the case $d_1 = 2$.

We prove by induction on $1 \le j \le d_1$ that

$$J_{d_2+d_3-2+j} = Shad(J_{d_2+d_3-3+j}) \cup \{x_3^{d_3+d_2-d_1+2j-2}\{x_1,x_2\}^{d_1-j}\}.$$

If $j = 1$ then $|J_{d_2+d_3-1}| - |Shad(J_{d_2+d_3-2})| = d_1$ so we must add d_1 generators, which are precisely the elements of the set $x_3^{d_3}\{x_1,x_2\}^{d-1}$ since J is strongly stable and x_3 is a strong Lefschetz element for S/J. The induction step is similar and finally we obtain that $J_{d_3+2d-2} = S_{d_3+2d-2}$ and thus we cannot add new minimal generators of J in degrees $> d_3 + 2d - 2$. $\square$

Corollary 2.2.9. *In the conditions of the above proposition, the number of minimal generators of J is $1 + d_1 + d_1 d_2$.*

Example 2.2.10. *Let $d_1 = 3$, $d_2 = 4$ and $d_3 = 9$. Then*

$$J = (x_1^3,\ x_1^2x_2^2,\ x_1x_2^4,\ x_2^6,\ x_2^5x_3,\ x_3^6x_2^3\{x_1,x_2\},$$
$$x_3^8x_2\{x_1,x_2\}^2,\ x_3^{10}\{x_1,x_2\}^2,\ x_3^{12}\{x_1,x_2\},\ x_3^{14}).$$

2.3 Case $d_1 + d_2 > d_3 + 1$.

- Subcase $d_1 = d_2 = d_3$.

Proposition 2.3.1. Let $2 \le d := d_1 = d_2 = d_3$ be positive integers. The Hilbert function of the standard graded complete intersection $A = K[x_1,x_2,x_3]/I$, where I is the ideal generated by f_1, f_2, f_3, with f_i homogeneous polynomials of degree d_i, for all i, with $1 \le i \le 3$, has the form:

1. $H(A,k) = \binom{k+2}{k}$, for $k \le d-1$.
2. $H(A,k) = \binom{k+2}{2} - \frac{3j(j+1)}{2}$, for $k = j + d - 1$, where $0 \le j \le \lfloor \frac{d-1}{2} \rfloor$.
3. $H(A,k) = H(A, 3d-k-3)$, for $k \ge \lceil \frac{3d-3}{2} \rceil$.

Proof. It follows from [31, Lemma 2.9(b)]. $\square$

Corollary 2.3.2. In the conditions of Proposition 2.3.1, let $J = Gin(I)$ be the generic initial ideal of I with respect to the reverse lexicographic order. If we denote by J_k the set of monomials of J of degree k, then:

1. $|J_k| = 0$, for $k \leq d-1$.
2. $|J_k| = \frac{3j(j+1)}{2}$, for $k = d-1+j$, where $0 \leq j \leq [\frac{3d-1}{2}]$.
3. If d is even, then $|J_k| = \frac{3d^2+3d(4j+2)}{8} + \frac{3j(j+1))}{2}$, for $k = j + \frac{3d-2}{2}$, where $0 \leq j \leq \frac{d-2}{2}$.
   If d is odd, then $|J_k| = \frac{3(d^2-1)+12jd}{8} + \frac{3j^2}{2}$, for $k = j + \frac{3d-3}{2}$, where $0 \leq j \leq \frac{d-1}{2}$.
4. $|J_k| = \frac{3d(d-1)}{2} + 3jd$, for $k = j + 2d - 2$, where $0 \leq j \leq d-1$.
5. $J_k = S_k$, for $k \geq 3d-2$.

Proposition 2.3.3. Let $2 \leq d := d_1 = d_2 = d_3$ be positive integers. Let $f_1, f_2, f_3 \in K[x_1, x_2, x_3]$ be a regular sequence of homogeneous polynomials of degrees d_1, d_2, d_3. If $I = (f_1, f_2, f_3)$, $J = Gin(I)$, the generic initial ideal with respect to the reverse lexicographic order, and S/I has (SLP), then:

$$J = (x_1^{d-2}\{x_1,x_2\}^2, x_1^{d-2j-1}x_2^{3j+1}, x_1^{d-2j-2}x_2^{3j+2}\ for\ 1 \leq j \leq \frac{d-3}{2},\ x_2^{\frac{3d-1}{2}}, x_3x_2^{\frac{3d-3}{3}},$$

$$x_3^{2j+1}x_1^{2j}x_2^{\frac{3d-3}{2}-3j}, \ldots, x_3^{2j+1}x_2^{\frac{3d-3}{2}-j}, 1 \leq j \leq \frac{d-3}{2}\ , x_3^{d-2+2j}\{x_1,x_2\}^{d-j}, 1 \leq j \leq d)$$

if d is odd, or

$$J = (x_1^{d-2}\{x_1,x_2\}^2, x_1^{d-2j-1}x_2^{3j+1}, x_1^{d-2j-2}x_2^{3j+2}\ for\ 1 \leq j \leq \frac{d-4}{2},\ x_1x_2^{\frac{3d-4}{2}}, x_2^{\frac{3d-2}{2}},$$

$$x_3^{2j}x_1^{2j-1}x_2^{\frac{3d}{2}-3j}, \ldots, x_3^{2j}x_2^{\frac{3d-2}{2}-j}, 1 \leq j \leq \frac{d-2}{2}, x_3^{d-2+2j}\{x_1,x_2\}^{d-j}, 1 \leq j \leq d)$$

if d is even.

Proof. We have $|J_d| = 3$, hence $J_d = \{x_1^{d-2}\{x_1,x_2\}^2\}$, since J is strongly stable and x_3 is strong Lefschetz for S/J. Therefore:

$$Shad(J_d) = \{x_1^{d-2}\{x_1,x_2\}^3, x_1^{d-2}x_3\{x_1,x_2\}^2\}.$$

Now we have four possibilities to analyze: $d = 2$, $d = 3$, $d = 4$ and $d \geq 5$.

d = 2. Using the formulas from Corrolary 2.3.2 we have $|J_3| - |Shad(J_2)| = 2$ so there are two generators to add to $Shad(J_2)$ to obtain J_3. Since J is strongly stable and x_3 is a strong Lefschetz element for S/J these new generators are $x_3^2x_1$ and $x_3^2x_2$. Therefore

$$J_3 = \{\{x_1,x_2\}^3, x_3\{x_1,x_2\}^2, x_3^2\{x_1,x_2\}\}.$$

Since $|J_4| - |Shad(J_3)| = 1$ there is only one generator to add to $Shad(J_3)$ and this is precisely x_3^4. It follows $J_4 = S_4$ and thus we cannot add new minimal generators of J in degree > 5.

$\mathbf{d=3}$. We have $|J_4| - |Shad(J_3)| = 2$ so there are two generators to add to $Shad(J_3)$ to obtain J_4. Since J is strongly stable and x_3 is a strong Lefschetz element for S/J these new generators are x_2^4 and $x_3x_2^3$. Therefore

$$J_4 = \{\{x_1, x_2\}^4, x_3\{x_1, x_2\}^3\}.$$

Since $|J_5| - |Shad(J_4)| = 3$ there are three new monomials to add to $Shad(J_4)$ in order to obtain J_5. Since J is strongly stable and x_3 is a strong Lefschetz element for S/J these new generators are $x_3^3\{x_1, x_2\}^2$. Analogously, we must add two new monomial to $Shad(J_5)$ in order to obtain J_6 and these are $x_3^5\{x_1, x_2\}$. Finally, we will add x_3^7 and thus we cannot add new minimal generators of J in degree > 7.

$\mathbf{d=4}$. We have $|J_4| - |Shad(J_3)| = 2$ so there are two generators to add to $Shad(J_4)$ to obtain J_5. Since J is strongly stable and x_3 is a strong Lefschetz element for S/J these new generators are $x_1x_2^4$ and x_2^5. Therefore

$$J_5 = \{\{x_1, x_2\}^5, x_1^2x_3\{x_1, x_2\}^2\}.$$

Since $|J_6| - |Shad(J_5)| = 2$ there are two new monomials to add to $Shad(J_5)$ in order to obtain J_6 and using the usual argument these new monomials are $x_3^2x_1x_2^2, x_3^2x_2^4$. It follows

$$J_6 = \{\{x_1, x_2\}^6, x_3\{x_1, x_2\}^5, x_3^2\{x_1, x_2\}^4\}.$$

Finally, we will add consequently $x_3^4\{x_1, x_2\}^3$, $x_3^6\{x_1, x_2\}^2$, $x_3^8\{x_1, x_2\}$ and x_3^{10}.

Suppose now $d \geq 5$. We have $|J_{d+1}| - |Shad(J_d)| = 2$ so there are two generators to add to $Shad(J_d)$ to obtain J_{d+1}. Since J is strongly stable and x_3 is a strong Lefschetz element for S/J these new generators are $x_1^{d-3}x_2^4, x_1^{d-4}x_2^5$. It follows

$$J_{d+1} = \{x_1^{d-4}\{x_1, x_2\}^5, x_1^{d-2}x_3\{x_1, x_2\}^3, x_1^{d-2}x_3^2\{x_1, x_2\}^2\}.$$

We prove by induction on j, with $1 \leq j \leq \left\lfloor \frac{d-3}{2} \right\rfloor$ that

$$J_{d+j} = Shad(J_{d+j-1}) \cup \{x_1^{d-2j-1}x_2^{3j+1}, x_1^{d-2j-2}x_2^{3j+2}\} =$$
$$= \{x_1^{d-2j-2}\{x_1, x_2\}^{3j+2}, x_1^{d-2j+1}x_3\{x_1, x_2\}^{3j-1}, \ldots, x_1^{d-2}x_3^j\{x_1, x_2\}^2\},$$

the assertion being checked for $j = 1$. Assume now that the statement is true for some $j \leq \left\lfloor \frac{d-3}{2} \right\rfloor$. Then

$$Shad(J_{d+j}) = \{x_1^{d-2j-2}\{x_1, x_2\}^{3j+3}, x_1^{d-2j-2}x_3\{x_1, x_2\}^{3j+2}, \ldots, x_1^{d-2}x_3^{j+1}\{x_1, x_2\}^2\}.$$

Since $|J_{d+j+1}| - |Shad(J_{d+j})| = 2$ we must add two generators to $Shad(J_{d+j})$ to obtain J_{d+j+1}. Using the fact that J is strongly stable and x_3 is a strong Lefschetz element for S/J it follows that these new generators are $x_1^{d-2j-3}x_2^{3j+4}, x_1^{d-2j-4}x_2^{3j+5}$, so the induction step is fulfilled.

We must consider now two possibilities.

1. d is odd. We obtain

$$J_{\frac{3d-3}{2}} = \{x_1\{x_1,x_2\}^{\frac{3d-5}{2}}, x_1^3x_3\{x_1,x_2\}^{\frac{3d-11}{2}}, \dots, x_1^{d-2}x_3^{\frac{d-3}{2}}\{x_1,x_2\}^2\}.$$

Since $|J_{\frac{3d-1}{2}}| - |Shad(J_{\frac{3d-3}{2}})| = 2$ there are two generators to add to $Shad(J_{\frac{3d-3}{2}})$ to obtain $J_{\frac{3d-1}{2}}$, and they must be $x_2^{\frac{3d-1}{2}}$, $x_3x_2^{\frac{3d-3}{3}}$ using the usual argument. Therefore,

$$J_{\frac{3d-1}{2}} = \{\{x_1,x_2\}^{\frac{3d-1}{2}}, x_3\{x_1,x_2\}^{\frac{3d-3}{2}}, \dots, x_1^{d-2}x_3^{\frac{d-1}{2}}\{x_1,x_2\}^2\}.$$

Since $|J_{\frac{3d+1}{2}}| - |Shad(J_{\frac{3d-1}{2}})| = 1$ we must add a 3 new generators to $Shad(J_{\frac{3d-1}{2}})$ to obtain $J_{\frac{3d+1}{2}}$ and since J is strongly stable and x_3 is a strong Lefschetz element for S/J, they are $x_1^2x_3^3x_2^{\frac{3d-9}{2}}, x_1x_3^3x_2^{\frac{3d-7}{2}}, x_3^3x_2^{\frac{3d-5}{2}}$.

We prove by induction on j, with $1 \le j \le \frac{d-3}{2}$ that

$$J_{\frac{3d-1}{2}+j} = \{Shad(J_{\frac{3d-3}{2}+j})\} \cup \{x_3^{2j+1}x_1^{2j}x_2^{\frac{3d-3}{2}-3j}, \dots, x_3^{2j+1}x_2^{\frac{3d-3}{2}-j}\} =$$

$$= \{\{x_1,x_2\}^{\frac{3d-1}{2}+j}, x_3\{x_1,x_2\}^{\frac{3d-3}{2}+j}, \dots, x_3^{2j+1}\{x_1,x_2\}^{\frac{3d-3}{2}-j},$$
$$x_3^{2j+2}x_1^{2j+3}\{x_1,x_2\}^{\frac{3d-11}{2}-3j}, \dots, x_1^{d-2}x_3^{\frac{d-1}{2}+j}\{x_1,x_2\}^2\}.$$

This assertion is proved for $j = 1$. Assume the assertion is true for some $j < \frac{d-3}{2}$. Since $|J_{\frac{3d+1}{2}+j}| - |Shad(J_{\frac{3d-1}{2}+j})| = 2j+3$ we must add $2j+3$ new generators to $Shad(J_{\frac{3d-1}{2}+j})$ in order to obtain $J_{\frac{3d+1}{2}+j}$. The usual argument implies that those new generators are $x_3^{2j+3}x_1^{2j+2}x_2^{\frac{3d-3}{2}-3j-3}, \dots, x_3^{2j+3}x_2^{\frac{3d-3}{2}-j-1}$, which conclude the induction.

2. d is even. We obtain

$$J_{\frac{3d-4}{2}} = \{x_1^2\{x_1,x_2\}^{\frac{3d-8}{2}}, x_1^4x_3\{x_1,x_2\}^{\frac{3d-14}{2}}, \dots, x_1^{d-2}x_3^{\frac{d-4}{2}}\{x_1,x_2\}^2\}.$$

We have $|J_{\frac{3d-2}{2}}| - |Shad(J_{\frac{3d-4}{2}})| = 2$ so we must add two new generators to $Shad(J_{\frac{3d-4}{2}})$ to obtain $J_{\frac{3d-2}{2}}$. Since J is strongly stable and x_3 is a strong Lefschetz element for S/J, they are $x_1x_2^{\frac{3d-4}{2}}$ and $x_2^{\frac{3d-2}{2}}$, therefore

$$J_{\frac{3d-2}{2}} = \{\{x_1,x_2\}^{\frac{3d-2}{2}}, x_1^2x_3\{x_1,x_2\}^{\frac{3d-8}{2}}, \dots, x_1^{d-2}x_3^{\frac{d-2}{2}}\{x_1,x_2\}^2\}.$$

Since $|J_{\frac{3d}{2}}| - |Shad(J_{\frac{3d-2}{2}})| = 2$ we must add two new generators to $Shad(J_{\frac{3d-2}{2}})$ in order to obtain $J_{\frac{3d}{2}}$ and since J is strongly stable and x_3 is a strong Lefschetz element for S/J, they are $x_1x_3^2x_2^{\frac{3d-6}{2}}$ and $x_3^2x_2^{\frac{3d-4}{2}}$.

We prove by induction on j, with $1 \le j \le \frac{d-2}{2}$ that

$$J_{\frac{3d-2}{2}+j} = Shad(J_{\frac{3d-4}{2}+j}) \cup \{x_3^{2j}x_1^{2j-1}x_2^{\frac{3d}{2}-3j}, \dots, x_3^{2j}x_2^{\frac{3d-2}{2}-j}\}.$$

The assertion has been proved for $j = 1$ and suppose it is true for some $j < \frac{d-2}{2}$. Since $|J_{\frac{3d}{2}+j}| - |Shad(J_{\frac{3d-2}{2}+j})| = 2j+2$ we must add $2j+2$ generators to $Shad(J_{\frac{3d-2}{2}+j})$ in order to obtain $J_{\frac{3d}{2}+j}$ and since J is strongly stable and x_3 is a strong Lefschetz element for S/J they must be $x_3^{2j+2}x_1^{2j+1}x_2^{\frac{3d}{2}-3j-3}, \dots, x_3^{2j+2}x_2^{\frac{3d-4}{2}-j}$, which conclude the induction.

Either if d is even, either if d is odd, we obtain

$$J_{2d-2} = \{\{x_1,x_2\}^{2d-2}, x_3\{x_1,x_2\}^{2d-2}, \dots, x_3^{d-2}\{x_1,x_2\}^d\} = \{\{x_1,x_2\}^d\{x_1,x_2,x_3\}^{d-2}\}.$$

Since $|J_{2d-1}| - |Shad(J_{2d-2})| = d$ we must add d new generators to $Shad(J_{2d-2})$ to obtain J_{2d-1}. But J is strongly stable and x_3 is a strong Lefschetz element for S/J, so we must add $x_3^d\{x_1,x_2\}^{d-1}$, therefore $J_{2d-1} = \{\{x_1,x_2\}^{d-1}\{x_1,x_2,x_3\}^d\}$. Using induction on $j \le d$, we prove that

$$J_{2d-2+j} = Shad(J_{2d-3+j}) \cup \{x_3^{d-2+2j}\{x_1,x_2\}^{d-j}\} = \{x_1,x_2\}^{d-j}\{x_1,x_2,x_3\}^{d-2+2j}.$$

For $j = 1$ we already proved. Suppose the assertion is true for $j < d$. We have $|J_{2d-1+j}| - |Shad(J_{2d-2+j})| = d-j$ so we must add $d-j$ new monomials to $Shad(J_{2d-2+j})$ to obtain J_{2d-1+j} and from the usual argument, these new monomials are $x_3^{d+2j}\{x_1,x_2\}^{d-j-1}$. Finally, since $J_{3d-2} = S_{3d-2}$ we cannot add new minimal generators of J in degree $> 3d-2$. $\square$

Corollary 2.3.4. *In the conditions above, the number of minimal generators of J is $1 + \frac{d(d+1)}{2} + \frac{(d+1)^2}{4}$ when d is odd, or $1 + \frac{d(d+1)}{2} + \frac{d(d+2)}{4}$ when d is even.*

Example 2.3.5. *1. Let $d_1 = d_2 = d_3 = 5$. Then*

$$J = (x_1^3\{x_1,x_2\}^2,\ x_1^2x_2^4,\ x_1x_2^5,\ x_2^7,\ x_3x_2^6,\ x_2^3x_3^3\{x_1,x_2\}^2,$$

$$x_3^5\{x_1,x_2\}^4,\ x_3^7\{x_1,x_2\}^3,\ x_3^9\{x_1,x_2\}^2,\ x_3^{11}\{x_1,x_2\},\ x_3^{13}).$$

2. Let $d_1 = d_2 = d_3 = 6$. Then

$$J = (x_1^4\{x_1,x_2\}^2,\ x_1^3x_2^4,\ x_1^2x_2^5,\ x_1x_2^7,\ x_2^8,\ x_2^6x_3^2\{x_1,x_2\},\ x_2^3x_3^4\{x_1,x_2\}^3,$$

$$x_3^6\{x_1,x_2\}^5,\ x_3^8\{x_1,x_2\}^4,\ x_3^{10}\{x_1,x_2\}^3,\ x_{12}^6\{x_1,x_2\}^2,\ x_{14}^6\{x_1,x_2\},\ x_{16}^6)$$

- Subcase $d_1 = d_2 < d_3$.

Proposition 2.3.6. Let $2 \le d := d_1 = d_2 < d_3$ be positive integers such that $d_1 + d_2 > d_3 + 1$. The Hilbert function of the standard graded complete intersection $A = K[x_1,x_2,x_3]/I$, where I is the ideal generated by f_1, f_2, f_3, with f_i homogeneous polynomials of degree d_i, for all i, with $1 \le i \le 3$, has the form:

1. $H(A,k) = \binom{k+2}{k}$, for $k \leq d-1$.

2. $H(A,k) = \binom{d+1}{2} + \sum_{i=1}^{j}(d-i)$, for $k = j+d-1$, where $0 \leq j \leq d_3 - d$.

3. $H(A,k) = \binom{d+1}{2} + \sum_{i=1}^{d_3-d}(d-i) + \sum_{i=1}^{j}(2d-d_3-2i)$, for $k = j+d_3-1$, where $0 \leq j \leq \lfloor \frac{2d-d_3-1}{2} \rfloor$.

4. $H(A,k) = H(A, d_3+2d-3-k)$,for $k \geq \lceil \frac{d_3+2d-3}{2} \rceil$.

Proof. It follows from [31, Lemma 2.9(b)]. □

Corollary 2.3.7. In the conditions of Proposition 2.3.6, let $J = Gin(I)$ be the generic initial ideal of I with respect to the reverse lexicographic order. If we denote by J_k the set of monomials of J of degree k, then:

1. $|J_k| = 0$, for $k \leq d-1$.

2. $|J_k| = j(j+1)$, for $k = j+d-1$ where $0 \leq j \leq d_3 - d$.

3. $|J_k| = d_3^2 + d_3 - d^2 - d - 2dd_3 + j(2d_3 - d) + \frac{3j(j+1)}{2}$, for $k = j + d_3 - 1$, where $0 \leq j \leq \lfloor \frac{2d-d_3-1}{2} \rfloor$.

4. If d_3 is even then $|J_k| = \frac{4d^2+3d_3^2-4dd_3+4d}{8} + \frac{j(2d+d_3)}{2} + \frac{3j(j+1)}{2}$, for $k = j + \frac{2d+d_3-2}{2}$, where $0 \leq j \leq \frac{2d-d_3-2}{2}$.

   If d_3 is odd then $|J_k| = \frac{3d_3^2+4d^2-4dd_3-3}{2} + \frac{j(2d+d_3-3)}{2} + \frac{3j(j+1)}{2}$, for $k = j + \frac{2d+d_3-3}{2}$, where $0 \leq j \leq \frac{2d-d_3-1}{2}$.

5. $|J_k| = 3d^2 - 2d + \frac{d_3(d_3+1)}{2} - 2dd_3 + (4d-d_3)j + j(j-1)$, for $k = j+2d-2$, where $0 \leq j \leq d_3 - d$.

6. $|J_k| = \frac{(d+d_3)(d+d_3-1)}{2} - \frac{d(d+1)}{2} + j(2d+d_3)$, for $k = j+d_3+d-2$, where $0 \leq j \leq d-1$.

7. $J_k = S_k$, for $k \geq 3d-2$.

Proposition 2.3.8. Let $2 \leq d := d_1 = d_2 < d_3$ be positive integers such that $2d > d_3 + 1$. Let $f_1, f_2, f_3 \in K[x_1, x_2, x_3]$ be a regular sequence of homogeneous polynomials of degrees d_1, d_2, d_3. If $I = (f_1, f_2, f_3)$, $J = Gin(I)$, the generic initial ideal with respect to the reverse lexicographic order and S/I has (SLP), then if d_3 is even, we have:

$$J = (x_1^d, x_1^{d-1}x_2, x_1^{d-j-1}x_2^{2j+1} \; for\ 1 \leq j \leq d_3 - d - 1,$$

$$x_1^{2d-d_3-2j+1}x_2^{2d_3-2d+3j-2}, x_1^{2d-d_3-2j}x_2^{2d_3-2d+3j-1} \; for\ 1 \leq j \leq \frac{2d-d_3}{2},$$

$$x_3^{2j}x_1^{2j-1}x_2^{\frac{2d+d_3-2}{2}-3j}, \ldots, x_3^{2j}x_2^{\frac{2d+d_3-4}{2}-j} \; for\ 1 \leq j \leq \frac{2d-d_3-2}{2},$$

$x_3^{2d-d_3-2+2j} x_2^{2d_3-2d+2-2j}\{x_1, x_2\}^{2d-d_3+j-2}, 1 \leq j \leq d_3 - d, x_3^{d_3-2+2j}\{x_1, x_2\}^{d-j}, 1 \leq j \leq d)$.

Otherwise, if d_3 is odd, we have

$$J = (x_1^d, x_1^{d-1}x_2, x_1^{d-j-1}x_2^{2j+1}\ for\ 1 \leq j \leq d_3 - d - 1,$$

$$x_1^{2d-d_3-2j+1}x_2^{2d_3-2d+3j-2}, x_1^{2d-d_3-2j}x_2^{2d_3-2d+3j-1}\ for\ 1 \leq j \leq \frac{2d-d_3-1}{2},$$

$$x_2^{\frac{2d+d_3-1}{2}}, x_3 x_2^{\frac{2d+d_3-3}{2}}, x_3^{2j+1}x_1^{2j}x_2^{\frac{2d+d_3-3}{2}-3j}, \ldots, x_3^{2j+1}x_2^{\frac{2d+d_3-3}{2}-j}, 1 \leq j \leq \frac{2d-d_3-3}{2},$$

$$x_3^{2d-d_3-2+2j}x_2^{2d_3-2d+2-2j}\{x_1, x_2\}^{2d-d_3+j-2}, 1 \leq j \leq d_3 - d, x_3^{d_3-2+2j}\{x_1, x_2\}^{d-j}, 1 \leq j \leq d).$$

Proof. We note first that $d \geq 3$. Indeed, if $d = 2$ then the condition $2d > d_3 + 1$ implies $d_3 = 2$ which is a contradiction. We have $|J_d| = 2$, hence $J_d = x_1^{d-1}\{x_1, x_2\}$, since J is strongly stable. Therefore:

$$Shad(J_d) = \{x_1^{d-1}\{x_1, x_2\}^2, x_1^{d-1}x_3\{x_1, x_2\}\}.$$

Assume $d_3 > d+1$. Since $|J_{d+1}| - |Shad(J_d)| = 1$ we must add a new generator to $Shad(J_d)$ in order to obtain J_{d+1}. On the other hand, J is strongly stable and x_3 is a strong Lefschetz element for S/J so this new generator is $x_1^{d-2}x_2^3$, therefore

$$J_{d+1} = \{x_1^{d-2}\{x_1, x_2\}^3, x_1^{d-1}x_3\{x_1, x_2\}\}.$$

We prove by induction on $1 \leq j \leq d_3 - d - 1$ that

$$J_{d+j} = Shad(J_{d-1+j}) \cup \{x_1^{d-j-1}x_2^{2j+1}\} = \{x_1^{d-j-1}\{x_1, x_2\}^{2j+1}, \ldots, x_1^{d-1}x_3^j\{x_1, x_2\}\}.$$

The case $j = 1$ was done. Suppose the assertion is true for some $j < d_3 - d - 1$. Then, since $|J_{d+j+1}| - |Shad(J_{d+j})| = 1$ it follows that we must add one generator to $Shad(J_{d+j})$ in order to obtain J_{d+j+1}. Since J is strongly stable and x_3 is a strong Lefschetz element for S/J, this new generator must be $x_1^{d-j-2}x_2^{2j+3}$. In particular,

$$J_{d_3-1} = \{x_1^{2d-d_3}\{x_1, x_2\}^{2d_3-2d-1}, x_1^{2d-d_3+1}x_3\{x_1, x_2\}^{2d_3-2d-3}, \ldots, x_3^{d_3-d-1}x_1^{d-1}\{x_1, x_2\}\},$$

which is the same formula as in the case $d_3 = d + 1$.

We need to consider several possibilities. First, suppose $d_3 = 2d - 2$. We have $|J_{d_3}| - |Shad(J_{d_3-1})| = 2$ so we must add two generators to $Shad(J_{d_3-1})$ in order to obtain $J_{d_3} = J_{2d-2}$. But since J is strongly stable and x_3 is a strong Lefschetz element for S/J, these new generators are $x_1x_2^{2d-3}$ and x_2^{2d-2}.

Suppose now $d_3 < 2d-2$. Since $|J_{d_3}| - |Shad(J_{d_3-1})| = 2$ we must add two generators to $Shad(J_{d_3-1})$ in order to obtain J_{d_3}. J strongly stable and x_3 is a strong Lefschetz element for S/J force us to choose $x_1^{2d-d_3-1}x_2^{2d_3-2d+1}, x_1^{2d-d_3-2}x_2^{2d_3-2d+2}$, so

$$J_{d_3} = \{x_1^{2d-d_3-2}\{x_1, x_2\}^{2d_3-2d}, x_1^{2d-d_3}x_3\{x_1, x_2\}^{2d_3-2d-1}, x_1^{2d-d_3+1}x_3^2\{x_1, x_2\}^{2d_3-2d-3}\}.$$

We show by induction on $1 \leq j \leq \lfloor \frac{2d-d_3+1}{2} \rfloor$ that

$$J_{d_3-1+j} = Shad(J_{d_3-2+j}) \cup \{x_1^{2d-d_3-2j+1}x_2^{2d_3-2d+3j-2}, x_1^{2d-d_3-2j}x_2^{2d_3-2d+3j-1}\} =$$
$$= \{x_1^{2d-d_3-2j}\{x_1,x_2\}^{2d_3-2d+3j-1}, x_3x_1^{2d-d_3-2j+2}\{x_1,x_2\}^{2d_3-2d+3j-4}, \dots,$$
$$x_3^jx_1^{2d-d_3}\{x_1,x_2\}^{2d_3-2d-1}, x_3^{j+1}x_1^{2d-d_3+1}\{x_1,x_2\}^{2d_3-2d-3}, \dots, x_3^{d_3-d-1+j}x_1^{d-1}\{x_1,x_2\}\}.$$

We already done the case $j = 1$. Suppose the assertion is true for some $j < \lfloor \frac{2d-d_3+1}{2} \rfloor$. We have $|J_{d_3+j}| - |Shad(J_{d_3+j-1})| = 2$ so we add two generators to $Shad(J_{d_3+j-1})$ and they must be $x_1^{2d-d_3-2j-1}x_2^{2d_3-2d+3j+1}, x_1^{2d-d_3-2j-2}x_2^{2d_3-2d+3j+2}$ from the usual argument. In the following, we distinguish between two possibilities: d is even or d is odd. If d is even, we get

$$J_{\frac{2d+d_3-4}{2}} = \{x_1^2\{x_1,x_2\}^{\frac{2d+d_3-8}{2}}, x_3x_1^4\{x_1,x_2\}^{\frac{2d+d_3-14}{2}}, \dots,$$
$$x_3^{\frac{2d-d_3-2}{2}}x_1^{2d-d_3}\{x_1,x_2\}^{2d_3-2d-1}, \dots, x_3^{\frac{d_3-3}{2}}x_1^{d-1}\{x_2,x_2\}\}.$$

We have $|J_{\frac{2d+d_3-2}{2}}| - |Shad(J_{\frac{2d+d_3-4}{2}})| = 2$ so we add two generators to $Shad(J_{\frac{2d+d_3-4}{2}})$ and they must be $x_1x_2^{\frac{2d+d_3-4}{2}}, x_2^{\frac{2d+d_3-2}{2}}$, so:

$$J_{\frac{2d+d_3-2}{2}} = \{\{x_1,x_2\}^{\frac{2d+d_3-2}{2}}, x_3x_1^2\{x_1,x_2\}^{\frac{2d+d_3-8}{2}}, \dots,$$
$$, x_3^{\frac{2d-d_3}{2}}x_1^{2d-d_3}\{x_1,x_2\}^{2d_3-2d-1}, \dots, x_3^{\frac{d_3-2}{2}}x_1^{d-1}\{x_2,x_2\}\}.$$

One can easily show by induction on $1 \leq j \leq \frac{2d-d_3-2}{2}$, if case, that

$$J_{\frac{2d+d_3-2}{2}+j} = Shad(J_{\frac{2d+d_3-4}{2}+j}) \cup \{x_3^{2j}x_1^{2j-1}x_2^{\frac{2d+d_3-2}{2}-3j}, \dots, x_3^{2j}x_2^{\frac{2d+d_3-4}{2}-j}\} =$$
$$= \{\{x_1,x_2\}^{\frac{2d+d_3-2}{2}+j}, \dots, x_3^{2j}\{x_1,x_2\}^{\frac{2d+d_3-2}{2}-j}, x_3^{2j+1}x_1^{2j+2}\{x_1,x_2\}^{\frac{2d+d_3-8}{2}-3j}, \dots,$$
$$x_3^{\frac{2d-d_3}{2}+j}x_1^{2d-d_3}\{x_1,x_2\}^{2d_3-2d-1}, \dots, x_3^{\frac{d_3-2}{2}+j}x_1^{d-1}\{x_1,x_2\}\}.$$

Indeed, if $j = 1$, we have $|J_{\frac{2d+d_3}{2}+j}| - |Shad(J_{\frac{2d+d_3-2}{2}+j})| = 2$ so we must add two monomials to $Shad(J_{\frac{2d+d_3-2}{2}+j})$ in order to obtain $J_{\frac{2d+d_3}{2}+j}$. Since J is strongly stable and x_3 is a strong Lefschetz element for S/J, these new monomials are $x_3^2x_1x_2^{\frac{2d+d_3-2}{2}-3}$ and $x_3^2x_2^{\frac{2d+d_3-2}{2}-2}$. The induction step is similar.

If d is odd, we have $J_{\frac{2d+d_3-3}{2}} = \{x_1\{x_1,x_2\}^{\frac{2d+d_3-5}{2}}, x_3x_1^3\{x_1,x_2\}^{\frac{2d+d_3-11}{2}}, \dots,$ $x_3^{\frac{2d-d_3-1}{2}}x_1^{2d-d_3}\{x_1,x_2\}^{2d_3-2d-1}, \dots, x_3^{\frac{d_3-3}{2}}x_1^{d-1}\{x_2,x_2\}\}$.

Since $|J_{\frac{2d+d_3-1}{2}}| - |Shad(J_{\frac{2d+d_3-3}{2}})| = 2$ we add two generators to $Shad(J_{\frac{2d+d_3-3}{2}})$ in order to obtain $J_{\frac{2d+d_3-1}{2}}$. Since J is strongly stable and x_3 is a strong Lefschetz element for S/J, these new monomials are $x_2^{\frac{2d+d_3-1}{2}}, x_3x_2^{\frac{2d+d_3-3}{2}}$, therefore:

$$J_{\frac{2d+d_3-1}{2}} = \{\{x_1,x_2\}^{\frac{2d+d_3-1}{2}}, x_3\{x_1,x_2\}^{\frac{2d+d_3-3}{2}}, x_3^2x_1^3\{x_1,x_2\}^{\frac{2d+d_3-11}{2}}, \dots,$$

$$x_3^{\frac{2d-d_3+1}{2}} x_1^{2d-d_3}\{x_1,x_2\}^{2d_3-2d-1}, \ldots, x_3^{\frac{d_3-1}{2}} x_1^{d-1}\{x_2,x_2\}\}.$$

Assume $d_3 < 2d-3$ (otherwise $\frac{2d+d_3-1}{2} = 2d-2$). $|J_{\frac{2d+d_3+1}{2}}| - |Shad(J_{\frac{2d+d_3-3}{2}})| = 3$ so we must add 3 generators to $Shad(J_{\frac{2d+d_3-3}{2}})$ to obtain $J_{\frac{2d+d_3+1}{2}}$. The usual argument implies that they are $x_1^2 x_3^3 x_2^{\frac{2d+d_3-9}{2}}, x_1 x_3^3 x_2^{\frac{2d+d_3-7}{2}}, x_3^3 x_2^{\frac{2d+d_3-5}{2}}$ and thus

$$J_{\frac{2d+d_3+1}{2}} = \{\{x_1,x_2\}^{\frac{2d+d_3+1}{2}}, x_3\{x_1,x_2\}^{\frac{2d+d_3-1}{2}}, x_3^2\{x_1,x_2\}^{\frac{2d+d_3-3}{2}}, x_3^3\{x_1,x_2\}^{\frac{2d+d_3-5}{2}}, \ldots,$$

$$x_3^{\frac{2d-d_3+3}{2}} x_1^{2d-d_3}\{x_1,x_2\}^{2d_3-2d-1}, \ldots, x_3^{\frac{d_3+1}{2}} x_1^{d-1}\{x_2,x_2\}\}.$$

One can easily prove by induction on $1 \le j \le \frac{2d-d_3-3}{2}$ that

$$J_{\frac{2d+d_3-1}{2}+j} = Shad(J_{\frac{2d+d_3-3}{2}+j}) \cup \{x_3^{2j+1} x_1^{2j} x_2^{\frac{2d+d_3-3}{2}-3j}, \ldots, x_3^{2j+1} x_2^{\frac{2d+d_3-3}{2}-j}\} =$$

$$= \{\{x_1,x_2\}^{\frac{2d+d_3-1}{2}+j}, \ldots, x_3^{2j+1}\{x_1,x_2\}^{\frac{2d+d_3-3}{2}-j}, x_3^{2j+2} x_1^{2j+3}\{x_1,x_2\}^{\frac{2d+d_3-11}{2}-3j},$$

$$x_3^{\frac{2d-d_3+1}{2}+j} x_1^{2d-d_3}\{x_1,x_2\}^{2d_3-2d-1}, \ldots, x_3^{\frac{d_3-1}{2}+j} x_1^{d-1}\{x_1,x_2\}\}.$$

In all cases above, we get: $J_{2d-2} = \{\{x_1,x_2\}^{2d-2}, x_3\{x_1,x_2\}^{2d-3}, \ldots, x_3^{2d-d_3-2}\{x_1,x_2\}^{d_3}, x_3^{2d-d_3-1} x_1^{2d-d_3}\{x_1,x_2\}^{2d_3-2d-1}, \ldots, x_3^{d-2} x_1^{d-1}\{x_1,x_2\}\}$.

We have $|J_{2d-1}| - |Shad(J_{2d-2})| = 2d-d_3$ so we must add $2d-d_3$ new generators to $Shad(J_{2d-2})$ to obtain J_{2d-1}. We get $J_{2d-1} = \{\{x_1,x_2\}^{2d-1}, x_3\{x_1,x_2\}^{2d-2}, \ldots, x_3^{2d-d_3}\{x_1,x_2\}^{d_3-1}, x_3^{2d-d_3+1} x_1^{2d-d_3+1}\{x_1,x_2\}^{2d_3-2d-3}, \ldots, x_3^{d-1} x_1^{d-1}\{x_1,x_2\}\}$.

One can easily show by induction of $1 \le j \le d_3 - d$ that

$$J_{2d-2+j} = Shad(J_{2d-3+j}) \cup \{x_3^{2d-d_3-2+2j} x_2^{2d_3-2d+2-2j}\{x_1,x_2\}^{2d-d_3+j-2}\} =$$

$$= \{\{x_1,x_2\}^{2d-2+j}, x_3\{x_1,x_2\}^{2d-3+j}, \ldots, x_3^{2d-d_3-2+2j}\{x_1,x_2\}^{d_3-j},$$
$$x_3^{2d-d_3-1+2j} x_1^{2d-d_3+j}\{x_1,x_2\}^{2d_3-2d-1-2j}, \ldots, x_3^{d-2+j} x_1^{d-1}\{x_1,x_2\}\}.$$

In particular, we get: $J_{d+d_3-2} = \{\{x_1,x_2\}^{d+d_3-2}, x_3\{x_1,x_2\}^{d+d_3-3}, \ldots, x_3^{d_3-2}\{x_1,x_2\}^{d}\}$. We have $|J_{d+d_3-1}| - |Shad(J_{d+d_3-2})| = d$ so we must add d new generators to $Shad(J_{d+d_3-2})$ in order to obtain J_{d+d_3-1}. Since J is strongly stable and x_3 is a strong Lefschetz element for S/J, these new generators are $x_3^{d_3}\{x_1,x_2\}^{d-1}$ so

$$J_{d+d_3-1} = \{\{x_1,x_2\}^{d+d_3-1}, x_3\{x_1,x_2\}^{d+d_3-2}, \ldots, x_3^{d_3}\{x_1,x_2\}^{d-1}\}.$$

Now, one can easily prove by induction on $1 \le j \le d-1$ that J_{d+d_3-2+j} is the set

$$Shad(J_{d+d_3-3+j}) \cup \{x_3^{d_3-2+2j}\{x_1,x_2\}^{d-j}\} = \{\{x_1,x_2\}^{d+d_3-2+j}, \ldots, x_3^{d_3-2+2j}\{x_1,x_2\}^{d-j}\}.$$

Finally we obtain that $J_{d_3+2d-2} = S_{d_3+2d-2}$ and thus we cannot add new minimal generators of J in degree $> d_3 + 2d - 2$. $\square$

Corollary 2.3.9. *In the conditions of the above proposition, the number of minimal generators of J is $d(d+1) - \left(\frac{2d-d_3}{2}\right)^2 + 1$ if d_3 is even or $d(d+1) - \frac{(2d-d_3)^2-1}{4} + 1$ if d_3 is odd.*

Example 2.3.10. *1. Let $d_1 = d_2 = 4$ and $d_3 = 6$. We have*

$$J = (x_1^4,\ x_1^3x_2,\ x_1^2x_2^3,\ x_1x_2^5,\ x_2^6,\ x_3^2x_2^4\{x_1,x_2\},\ x_3^4x_2^2\{x_1,x_2\}^2,$$

$$x_3^6\{x_1,x_2\}^3,\ x_3^8\{x_1,x_2\}^2,\ x_3^{10}\{x_1,x_2\},\ x_3^{12}).$$

2. Let $d_1 = d_2 = 4$ and $d_3 = 5$. We have:

$$J = (x_1^4,\ x_1^3x_2,\ x_1^2x_2^3,\ x_1x_2^4,\ x_2^6,\ x_2^5x_3,\ x_3^3x_2^2\{x_1,x_2\}^2,$$

$$x_3^5\{x_1,x_2\}^3,\ x_3^7\{x_1,x_2\}^2,\ x_3^9\{x_1,x_2\},\ x_3^{11}).$$

- Subcase $d_1 < d_2 = d_3$.

Proposition 2.3.11. Let $2 \le d_1 < d_2 = d_3 =: d$ be positive integers. The Hilbert function of the standard graded complete intersection $A = K[x_1,x_2,x_3]/I$, where I is the ideal generated by $f_1,\ f_2,\ f_3$, with f_i homogeneous polynomials of degree d_i, for all i, with $1 \le i \le 3$, has the form:

1. $H(A,k) = \binom{k+2}{k}$, for $k \le d_1 - 2$.
2. $H(A,k) = \binom{d_1+1}{2} + jd_1$, for $k = j + d_1 -$, where $0 \le j \le d - d_1$.
3. $H(A,k) = \binom{d_1+1}{2} + d_1(d-d_1) + \sum_{i=1}^{j}(d_1 - 2i)$, for $k = j + d - 1$, where $0 \le j \le \left\lfloor \frac{d_1-1}{2} \right\rfloor$.
4. $H(A,k) = H(A, d_1 + 2d - 3 - k)$, for $k \ge \left\lceil \frac{d_1+2d-3}{2} \right\rceil$.

Proof. It follows from [31, Lemma 2.9(b)]. □

Corollary 2.3.12. In the conditions of Proposition 2.3.11, let $J = Gin(I)$ be the generic initial ideal of I with respect to the reverse lexicographic order. If we denote by J_k the set of monomials of J of degree k, then:

1. $|J_k| = 0$, for $k \le d_1 - 1$.
2. $|J_k| = j(j+1)/2$, for $k = j + d_1 - 1$, where $0 \le j \le d - d_1$.
3. $|J_k| = \frac{(d-d_1)(d-d_1-1)}{2} + j(d-d_1) + \frac{3j(j+1)}{2}$, for $k = j + d - 1$, where $0 \le j \le \left\lfloor \frac{d_1-1}{2} \right\rfloor$.

4. If d_1 is even then $|J_k| = \frac{3d_1^2+2d_1+4d^2+4d-4dd_1}{8} + \frac{j(2d+d_1)}{2} + \frac{3j(j+1)}{2}$, for $k = j + \frac{2d+d_1-2}{2}$, where $0 \leq j \leq \frac{d_1-2}{2}$.

   If d_1 is odd then $|J_k| = \frac{3d_1^2+4d^2-4dd_1-3}{2} + \frac{j(2d+d_1)}{2} + \frac{3j^2}{2}$ for $k = j + \frac{2d+d_1-3}{2}$, where $0 \leq j \leq \frac{d_1-1}{2}$.

5. $|J_k| = \frac{d(d-1)}{2} + d_1(d_1-1) + j(2d_1+d) + \frac{j(j-1)}{2}$, for $k = j + d_1 + d - 2$, where $0 \leq j \leq d - d_1$.

6. $|J_k| = \frac{2d(2d-1)-d_1(d_1-1)}{2} + j(2d+d_1))$, for $k = j + 2d - 2$, where $0 \leq j \leq d_1 - 1$.

7. $J_k = S_k$, for $k \geq 3d - 2$.

Proposition 2.3.13. Let $2 \leq d_1 < d_2 = d_3 =: d$ be positive integers. Let $f_1, f_2, f_3 \in K[x_1, x_2, x_3]$ be a regular sequence of homogeneous polynomials of degrees d_1, d_2, d_3. Let $I = (f_1, f_2, f_3)$, $J = Gin(I)$, the generic initial ideal with respect to the reverse lexicographic order and S/I has (SLP), then if d_1 is even we have:

$$J = (x_1^{d_1}, x_1^{d_1-2j+1}x_2^{d-d_1-2+3j}, x_1^{d_1-2j}x_2^{d-d_1-1+3j} \; for \; 1 \leq j \leq \frac{d_1-2}{2},$$

$$x_1x_2^{\frac{d_1+2d-4}{2}}, x_2^{\frac{d_1+2d-4}{2}}, x_3^{2j}x_1^{2j-1}x_2^{\frac{d_1+2d}{2}-3j}, \ldots, x_3^{2j}x_2^{\frac{d_1+2d-2}{2}-j} \; for \; 1 \leq j \leq \frac{d_1-4}{2},$$

$$x_3^{d_1-2+2j}x_2^{d-d_1-j+1}\{x_1, x_2\}^{d_1-1} \; for \; 1 \leq j \leq d - d_1,$$

$$x_3^{2d-d_1-2+2j}\{x_1, x_2\}^{d_1-j} \; for \; 1 \leq j \leq d_1).$$

Otherwise, if d_1 is odd, then:

$$J = (x_1^{d_1}, x_1^{d_1-2j+1}x_2^{d-d_1-2+3j}, x_1^{d_1-2j}x_2^{d-d_1-1+3j} \; for \; 1 \leq j \leq \frac{d_1-1}{2},$$

$$x_2^{\frac{d_1+2d-1}{2}}, x_3x_2^{\frac{d_1+2d-3}{2}}, x_3^{2j+1}x_1^{2j}x_2^{\frac{d_1+2d-3}{2}-3j}, \ldots, x_3^{2j+1}x_2^{\frac{d_1+2d-3}{2}-3j} \; for \; 1 \leq j \leq \frac{d_1-3}{2},$$

$$x_3^{d_1-2+2j}x_2^{d-d_1-j+1}\{x_1, x_2\}^{d_1-1} \; for \; 1 \leq j \leq d - d_1,$$

$$x_3^{2d-d_1-2+2j}\{x_1, x_2\}^{d_1-j} \; for \; 1 \leq j \leq d_1).$$

Proof. We have $|J_{d_1}| = 1$, hence $J_{d_1} = \{x_1^{d_1}\}$, since J is a strongly stable. Therefore:

$$Shad(J_{d_1}) = \{x_1^{d_1}\{x_1, x_2\}, x_1^{d_1}x_3\}.$$

Assume $d > d_1 + 1$. Using the formulas from 2.3.12 we get $|J_{d_1+1}| - |Shad(J_{d_1})| = 0$ so $J_{d_1+1} = Shad(J_{d_1})$. We show by induction on $1 \leq j \leq d - d_1$ that

$$J_{d_1+j} = Shad(J_{d_1-1+j}) = \{x_1^{d_1}\{x_1, x_2\}^j, x_1^{d_1}x_3\{x_1, x_2\}^{j-1}, \ldots, x_1^{d_1}x_3^j\}.$$

We already prove this for $j=1$. Suppose the assertion is true for some $j<d-d_1$. Since $|J_{d_1+j}|=|Shad(J_{d_1-1+j})|$ we have $J_{d_1+j}=Shad(J_{d_1-1+j})$ thus the induction step is done. In particular, we get:

$$J_{d-1}=\{x_1^{d_1}\{x_1,x_2\}^{d-d_1-1},x_1^{d_1}x_3\{x_1,x_2\}^{d-d_1-2},\ldots,x_1^{d_1}x_3^{d-d_1-1}\}.$$

which is the same expression as in the case $d=d_1+1$.

In the following, we consider two possibilities. First, suppose $d_1=2$. We have $|J_d|-|Shad(J_{d-1})|=2$ so we must add two new generators to $Shad(J_{d-1})$ in order to obtain J_d. Since J is strongly stable and x_3 is a strong Lefschetz element for S/J these new generators are $x_1x_2^{d-1}$ and x_2^d.

Suppose now $d_1>2$. We have $|J_d|-|Shad(J_{d-1})|=2$ so we must add two new generators to $Shad(J_{d-1})$ in order to obtain J_d. Since J is strongly stable and x_3 is a strong Lefschetz element for S/J these new generators are $x_1^{d_1-1}x_2^{d-d_1+1},x_1^{d_1-2}x_2^{d-d_1+2}$. Therefore

$$J_d=\{x_1^{d_1-2}\{x_1,x_2\}^{d-d_1},x_1^{d_1}x_3\{x_1,x_2\}^{d-d_1-1},\ldots,x_1^{d_1}x_3^{d-d_1}\}.$$

We prove by induction on $1\leq j\leq\left\lfloor\frac{d_1-1}{2}\right\rfloor$ that:

$$J_{d-1+j}=Shad(J_{d-2+j})\cup\{x_1^{d_1-2j+1}x_2^{d-d_1-2+3j},x_1^{d_1-2j}x_2^{d-d_1-1+3j}\}=$$

$$=\{x_1^{d_1-2j}\{x_1,x_2\}^{d-d_1-1-3j},x_1^{d_1-2j+2}x_3\{x_1,x_2\}^{d-d_1-4-3j},\ldots,x_1^{d_1}x_3^{j}\{x_1,x_2\}^{d-d_1-1},$$
$$x_1^{d_1}x_3^{j+1}\{x_1,x_2\}^{d-d_1-2},\ldots,x_1^{d_1}x_3^{d-d_1+j-1}\}.$$

We already done the case $j=1$. Suppose the assertion is true for some $j<\left\lfloor\frac{d_1-1}{2}\right\rfloor$. Since $|J_{d+j}|-|Shad(J_{d-1+j})|=2$ we must add two generators to $Shad(J_{d-1+j})$ in order to obtain $|J_{d+j}|$ and they are $x_1^{d_1-2j-1}x_2^{d-d_1+3j+1},x_1^{d_1-2j-2}x_2^{d-d_1+3j+2}$ because J is strongly stable and x_3 is a strong Lefschetz element for S/J. Therefore, the induction step is done.

In the following, we consider two possibilities: d_1 is even or d_1 is odd. First, suppose d_1 is even. We have

$$J_{\frac{d_1+2d-4}{2}}=\{x_1^2\{x_1,x_2\}^{\frac{d_1+2d-8}{2}},x_1^4x_3\{x_1,x_2\}^{\frac{d_1+2d-14}{2}},\ldots,x_1^{d_1}x_3^{\frac{d_1-2}{2}}\{x_1,x_2\}^{d-d_1-1},$$

$$x_1^{d_1}x_3^{\frac{d_1}{2}}\{x_1,x_2\}^{d-d_1-2},\ldots,x_1^{d_1}x_3^{\frac{2d-d_1-4}{2}}\}.$$

Since $|J_{\frac{d_1+2d-2}{2}}|-|Shad(J_{\frac{d_1+2d-4}{2}})|=2$ we need to add two new monomials to $Shad(J_{\frac{d_1+2d-4}{2}})$ and since J is strongly stable and x_3 is a strong Lefschetz element for S/J, they are $x_1x_2^{\frac{d_1+2d-4}{2}},x_2^{\frac{d_1+2d-4}{2}}$, thus:

$$J_{\frac{d_1+2d-2}{2}}=\{\{x_1,x_2\}^{\frac{d_1+2d-2}{2}},x_3x_1^2\{x_1,x_2\}^{\frac{d_1+2d-8}{2}},x_1^4x_3^2\{x_1,x_2\}^{\frac{d_1+2d-14}{2}},\ldots,$$

$$x_1^{d_1}x_3^{\frac{d_1}{2}}\{x_1,x_2\}^{d-d_1-1},x_1^{d_1}x_3^{\frac{d_1+2}{2}}\{x_1,x_2\}^{d-d_1-2},\ldots,x_1^{d_1}x_3^{\frac{2d-d_1-2}{2}}\}.$$

One can easily show by induction on $1 \le j \le \frac{d_1-2}{2}$ that

$$J_{\frac{d_1+2d-2}{2}+j} = Shad(J_{\frac{d_1+2d-4}{2}+j}) \cup \{x_3^{2j}x_1^{2j-1}x_2^{\frac{d_1+2d}{2}-3j}, \ldots, x_3^{2j}x_2^{\frac{d_1+2d-2}{2}-j}\} =$$

$$= \{\{x_1,x_2\}^{\frac{d_1+2d-2}{2}+j}, x_3\{x_1,x_2\}^{\frac{d_1+2d-4}{2}+j}, \ldots, x_3^{2j}\{x_1,x_2\}^{\frac{d_1+2d-2}{2}-j},$$

$$x_3^{2j+1}x_1^{2j+2}\{x_1,x_2\}^{\frac{d_1+2d-8}{2}-3j}, \ldots, x_3^{\frac{d_1}{2}}x_1^{d_1}\{x_1,x_2\}^{d-d_1-1}, \ldots, x_1^{d_1}x_3^{\frac{2d-d_1-2}{2}+j}\}.$$

The assertion was already done for $j = 1$ and the induction step is similar.

If d_1 is odd, we get

$$J_{\frac{d_1+2d-3}{2}} = \{x_1\{x_1,x_2\}^{\frac{d_1+2d-5}{2}}, x_1^3x_3\{x_1,x_2\}^{\frac{d_1+2d-11}{2}}, \ldots, x_1^{d_1}x_3^{\frac{d_1-1}{2}}\{x_1,x_2\}^{d-d_1-1},$$

$$x_1^{d_1}x_3^{\frac{d_1+1}{2}}\{x_1,x_2\}^{d-d_1-2}, \ldots, x_1^{d_1}x_3^{\frac{2d-d_1-3}{2}}\}.$$

Since $|J_{\frac{d_1+2d-1}{2}}| - |Shad(J_{\frac{d_1+2d-3}{2}})| = 2$ we add two generators to $Shad(J_{\frac{d_1+2d-3}{2}})$ in order to obtain $J_{\frac{d_1+2d-1}{2}}$ and they must be $x_2^{\frac{d_1+2d-1}{2}}, x_3x_2^{\frac{d_1+2d-3}{2}}$, therefore:

$$J_{\frac{d_1+2d-1}{2}} = \{\{x_1,x_2\}^{\frac{d_1+2d-1}{2}}, x_3\{x_1,x_2\}^{\frac{d_1+2d-3}{2}}, x_1^3x_3^2\{x_1,x_2\}^{\frac{d_1+2d-11}{2}}, \ldots,$$

$$x_1^{d_1}x_3^{\frac{d_1+1}{2}}\{x_1,x_2\}^{d-d_1-1}, x_1^{d_1}x_3^{\frac{d_1+3}{2}}\{x_1,x_2\}^{d-d_1-2}, \ldots, x_1^{d_1}x_3^{\frac{2d-d_1-1}{2}}\}.$$

Since $|J_{\frac{d_1+2d+1}{2}}| - |Shad(J_{\frac{d_1+2d-1}{2}})| = 3$, we add 3 new generators to $Shad(J_{\frac{d_1+2d-1}{2}})$ in order to obtain $J_{\frac{d_1+2d+1}{2}}$ and they are $x_1^2x_3^3x_2^{\frac{d_1+2d-9}{2}}, x_1x_3^3x_2^{\frac{d_1+2d-7}{2}}, x_3^3x_2^{\frac{d_1+2d-5}{2}}$, therefore

$$J_{\frac{d_1+2d+1}{2}} = \{\{x_1,x_2\}^{\frac{d_1+2d+1}{2}}, x_3\{x_1,x_2\}^{\frac{d_1+2d-1}{2}}, x_3^2\{x_1,x_2\}^{\frac{d_1+2d-3}{2}}, x_3^3\{x_1,x_2\}^{\frac{d_1+2d-5}{2}}, \ldots,$$

$$x_1^{d_1}x_3^{\frac{d_1+3}{2}}\{x_1,x_2\}^{d-d_1-1}, x_1^{d_1}x_3^{\frac{d_1+5}{2}}\{x_1,x_2\}^{d-d_1-2}, \ldots, x_1^{d_1}x_3^{\frac{2d-d_1+1}{2}}\}.$$

One can easily prove by induction on $1 \le j \le \frac{d_1-3}{2}$ that

$$J_{\frac{d_1+2d-1}{2}+j} = Shad(J_{\frac{d_1+2d-3}{2}+j}) \cup \{x_3^{2j+1}x_1^{2j}x_2^{\frac{d_1+2d-3}{2}-3j}, \ldots, x_3^{2j+1}x_2^{\frac{d_1+2d-3}{2}-3j}\} =$$

$$= \{\{x_1,x_2\}^{\frac{d_1+2d-1}{2}+j}, x_3\{x_1,x_2\}^{\frac{d_1+2d-3}{2}+j}, \ldots, x_3^{2j+2}\{x_1,x_2\}^{\frac{d_1+2d-3}{2}-j},$$

$$x_3^{2j+2}x_1^{2j+3}\{x_1,x_2\}^{\frac{d_1+2d-11}{2}-3j}, \ldots, x_3^{\frac{d_1+1}{2}}x_1^{d_1}\{x_1,x_2\}^{d-d_1-1}, \ldots, x_1^{d_1}x_3^{\frac{2d-d_1-1}{2}+j}\}.$$

The assertion was already proved for $j = 1$ and the induction step is similar.

In all cases, we obtain

$$J_{d_1+d-2} = \{\{x_1,x_2\}^{d_1+d-2}, x_3\{x_1,x_2\}^{d_1+d-3}, \ldots, x_3^{d_1-2}\{x_1,x_2\}^{d},$$

$$x_1^{d_1}x_3^{d_1-1}\{x_1,x_2\}^{d-d_1-1},\dots,x_1^{d_1}x_3^{d-2}\}.$$

We have $|J_{d_1+d-1}|-|Shad(J_{d_1+d-2})|=d_1$ so we must add d_1 new monomials to $Shad(J_{d_1+d-2})$ in order to obtain J_{d_1+d-1}. Since J is strongly stable and x_3 is a strong Lefschetz element for S/J, these new monomials are $x_3^{d_1}x_2^{d-d_1}\{x_1,x_2\}^{d_1-1}$, therefore

$$J_{d_1+d-1}=(\{x_1,x_2\}^{d_1+d-1},x_3\{x_1,x_2\}^{d_1+d-2},\dots,x_3^{d_1}\{x_1,x_2\}^{d-1},$$
$$x_1^{d_1}x_3^{d_1+1}\{x_1,x_2\}^{d-d_1-2},\dots,x_1^{d_1}x_3^{d-1}).$$

One can easily prove by induction on $1\le j\le d-d_1$ that

$$J_{d_1+d-1+j}=Shad(J_{d_1+d-2+j})\cup\{x_3^{d_1-2+2j}x_2^{d-d_1-j+1}\{x_1,x_2\}^{d_1-1}\}=$$
$$=\{\{x_1,x_2\}^{d_1+d-2+j},x_3\{x_1,x_2\}^{d_1+d-3+j},\dots,x_3^{d_1-2+2j}\{x_1,x_2\}^{d-j},$$
$$x_1^{d_1}x_3^{d_1-1+2j}\{x_1,x_2\}^{d-d_1-j-1},\dots,x_1^{d_1}x_3^{d-2+j}\}$$

the case $j=1$ being already done and than, the induction step being similar. In particular, $J_{2d-2}=\{\{x_1,x_2\}^{2d-2},x_3\{x_1,x_2\}^{2d-3},\dots,x_3^{2d-d_1-2}\{x_1,x_2\}^{d_1}\}$.

We have $|J_{2d-1}|-|Shad(J_{2d-2})|=d_1$ so we must add d_1 new generators to $Shad(J_{2d-2})$ in order to obtain J_{2d-1}. Since J is strongly stable and x_3 is a strong Lefschetz element for S/J, these new monomials are $x_3^{2d-d_1}\{x_1,x_2\}^{d_1-1}$ so

$$J_{2d-1}=\{\{x_1,x_2\}^{2d-1},x_3\{x_1,x_2\}^{2d-2},\dots,x_3^{2d-d_1}\{x_1,x_2\}^{d_1-1}\}.$$

One can easily show by induction on $1\le j\le d_1$ that

$$J_{2d-2+j}=Shad(J_{2d-3+j})\cup\{x_3^{2d-d_1-2+2j}\{x_1,x_2\}^{d_1-j}\}=$$
$$=\{\{x_1,x_2\}^{2d-2+j},\dots,x_3^{2d-d_1-2+2j}\{x_1,x_2\}^{d_1-j}\}.$$

Finally, we obtain $J_{d_1+2d-2}=S_{d_1+2d-2}$ and therefore we cannot add new minimal generators of J in degrees $>d_1+2d-2$. $\square$

Corollary 2.3.14. *In the conditions of the above proposition, the number of minimal generators of J is $d_1(d+1)-\left(\frac{d_1}{2}\right)^2+1$ if d is even; $d_1(d+1)-\frac{d_1^2-1}{4}+1$ if d is odd.*

Example 2.3.15. *If $d_1=4$, $d_2=d_3=6$, then $J=(x_1^4,\ x_1^3x_2^3,\ x_1^2x_2^4,\ x_2^7,\ x_3x_2^6,\ x_3^2x_1x_2^5,\ x_3^2x_2^6,\ x_3^4x_2^2\{x_1,x_2\}^3,\ x_3^6x_2\{x_1,x_2\}^3,\ x_3^8\{x_1,x_2\}^3,\ x_3^{10}\{x_1,x_2\}^2,\ x_3^{12}\{x_1,x_2\},\ x_3^{14})$.*

If $d_1=3$ and $d_2=d_3=6$, then: $J=(x_1^3,\ x_1^2x_2^4,\ x_1x_2^5,\ x_1x_2^6,\ x_2^7,\ x_3^3x_2^3\{x_1,x_2\}^2,\ x_3^5x_2^2\{x_1,x_2\}^2,\ x_3^7x_2\{x_1,x_2\}^2,\ ,x_3^9\{x_1,x_2\}^2,\ x_3^{11}\{x_1,x_2\}'\ x_3^{11})$.

- Subcase $d_1<d_2<d_3$.

Proposition 2.3.16. Let $2 \leq d_1 < d_2 < d_3$ be positive integers such that $d_1 + d_2 > d_3 + 1$. The Hilbert function of the standard graded complete intersection $A = K[x_1, x_2, x_3]/I$, where I is the ideal generated by f_1, f_2, f_3, with f_i homogeneous polynomials of degree d_i, for all i, with $1 \leq i \leq 3$, has the form:

1. $H(A, k) = \binom{k+2}{k}$, for $k \leq d_1 - 2$.

2. $H(A, k) = \binom{d_1+1}{2} + jd_1$, for $k = j + d_1 - 1$, where $0 \leq j \leq d_2 - d_1$.

3. $H(A, k) = \binom{d_1+1}{2} + d_1(d_2 - d_1) + \sum_{i=1}^{j}(d_1 - i)$, for $k = j + d_2 - 1$, where $0 \leq k \leq d_3 - d_2$.

4. $H(A, k) = \binom{d_1+1}{2} + d_1(d_2 - d_1) + \sum_{i=1}^{d_3-1}(d_1 - j) + \sum_{i=1}^{j}(d_1 + d_2 - d_3 - 2i)$, for $k = j + d_3 - 1$, where $0 \leq j \leq [\frac{d_1+d_2-d_3-1}{2}]$.

5. $H(A, k) = H(A, d_1 + d_2 + d_3 - 3 - k)$ for $k > d_3 - 1 + [\frac{d_1+d_2-d_3-1}{2}]$.

Proof. It follows from [31, Lemma 2.9(b)]. □

Corollary 2.3.17. In the conditions of Proposition 2.3.16, let $J = Gin(I)$ be the generic initial ideal of I with respect to the reverse lexicographic order. If we denote by J_k the set of monomials of J of degree k, then:

1. $|J_k| = 0$, for $k \leq d_1 - 1$.

2. $|J_k| = j(j + 1)/2$, for $k = j + d_1 - 1$, where $0 \leq j \leq d_2 - d_1$.

3. $|J_k| = d_2(d_2 - 1) + j(d_2 - d_1) + j(j + 1)$, for $k = j + d_2 - 1$, where $0 \leq j \leq d_3 - d_2$.

4. $|J_k| = |J_{d_3-1}| + j(2d_3 - d_1 - d_2) + \frac{3j(j+1)}{2}$, for $k = j + d_3 - 1$, where $0 \leq j \leq [\frac{\alpha-1}{2}]$.

5. If $d_1 + d_2 + d_3$ is even then $|J_k| = |J_{\frac{d_1+d_2+d_3-4}{2}}| + \frac{(j+1)(d_1+d_2+d_3)}{2} + \frac{3j(j+1)}{2}$, for $k = j + \frac{d_1+d_2+d_3-2}{2}$, where $0 \leq j \leq \frac{d_1+d_2-d_3-2}{2}$.

   If $d_1 + d_2 + d_3$ is odd then $|J_k| = |J_{\frac{d_1+d_2+d_3-3}{2}}| + \frac{j(d_1+d_2+d_3)}{2} + \frac{3j^2}{2}$, for $k = j + \frac{d_1+d_2+d_3-3}{2}$, where $0 \leq j \leq \frac{d_1+d_2-d_3-1}{2}$.

6. $|J_k| = |J_{d_1+d_2-2}| + j(2d_1 + 2d_2 - d_3 - 1) + j^2$, for $k = j + d_1 + d_2 - 2$, where $0 \leq d_3 - d_2$.

7. $|J_k| = |J_{d_1+d_3-2}| + j(2d_1 + d_3 - 1) + \frac{j(j+1)}{2}$, for $k = j + d_1 + d_3 - 2$, where $0 \leq j \leq d_2 - d_1$.

8. $|J_k| = |J_{d_2+d_3-1}| + j(d_1 + d_2 + d_3)$, for $k = j + d_1 + d_3 - 2$, where $0 \leq j \leq d_1 - 1$

9. $J_k = S_k$, for $k \geq 3d - 2$.

Proposition 2.3.18. Let $2 \leq d_1 < d_2 < d_3$ be positive integers such that $d_1 + d_2 > d_3 + 1$. Let $f_1, f_2, f_3 \in K[x_1, x_2, x_3]$ be a regular sequence of homogeneous polynomials of degrees d_1, d_2, d_3. Let $\alpha = d_1 + d_2 - d_3$. Let $I = (f_1, f_2, f_3)$, $J = Gin(I)$, the generic initial ideal with respect to the reverse lexicographic order, and suppose S/I has (SLP). If α is even, then:

$$J = (x_1^{d_1}, x_1^{d_1-1}x_2^{d_2-d_1+1}, x_1^{d_1-2}x_2^{d_2-d_1+3}, \ldots, x_1^{d_1+d_2-d_3}x_2^{2d_3-d_1-d_2-1},$$

$$x_1^{d_1+d_2-d_3-2j}x_2^{2d_3-d_1-d_2+3j-1}, x_1^{d_1+d_2-d_3-2j+1}x_2^{2d_3-d_1-d_2+3j-2} for\ j = 1, \ldots, \frac{\alpha-2}{2},$$

$$x_2^{\frac{d_1+d_2+d_3-2}{2}}, x_1x_2^{\frac{d_1+d_2+d_3-4}{2}}, x_3^{2j}x_2^{\frac{d_1+d_2+d_3}{2}-3j}\{x_1, x_2\}^{2j-1} for\ j = 1, \ldots, \frac{\alpha-2}{2},$$

$$x_1^{d_1+d_2-d_3+j-2}x_2^{2d_3-d_1-d_2-2j+2}x_3^{d_1+d_2-d_3+2j-2}, .., x_2^{d_3-j}x_3^{d_1+d_2-d_3+2j-2}\ for\ j = 1, .., d_3 - d_2,$$

$$x_1^{d_1-1}x_3^{d_1+d_3-d_2+2j-2}x_2^{d_2-d_1-j+1}, \ldots, x_3^{d_1+d_3-d_2+2j-2}x_2^{d_2-j}\ for\ j = 1, \ldots, d_2 - d_1,$$

$$\{x_1, x_2\}^{d_1-j}x_3^{d_2+d_3-d_1-2+2j}\ for\ j = 1, \ldots, d_1).$$

Otherwise, if α is odd, then:

$$J = (x_1^{d_1}, x_1^{d_1-1}x_2^{d_2-d_1+1}, x_1^{d_1-2}x_2^{d_2-d_1+3}, \ldots, x_1^{d_1+d_2-d_3}x_2^{2d_3-d_1-d_2-1},$$

$$x_1^{d_1+d_2-d_3-2j}x_2^{2d_3-d_1-d_2+3j-1}, x_1^{d_1+d_2-d_3-2j+1}x_2^{2d_3-d_1-d_2+3j-2}, j = 1, \ldots, \frac{\alpha-1}{2},$$

$$x_2^{\frac{d_1+d_2+d_3-1}{2}}, x_3x_2^{\frac{d_1+d_2+d_3-3}{2}}, x_1^{2j}x_3^{2j+1}x_2^{\frac{d_1+d_2+d_3-3}{2}-3j}, .., x_3^{2j+1}x_2^{\frac{d_1+d_2+d_3-3}{2}-j}, j = 1, .., \frac{\alpha-3}{2},$$

$$x_1^{d_1+d_2-d_3+j-2}x_2^{2d_3-d_1-d_2-2j+2}x_3^{d_1+d_2-d_3+2j-2}, .., x_2^{d_3-j}x_3^{d_1+d_2-d_3+2j-2}\ for\ j = 1, .., d_3 - d_2,$$

$$x_1^{d_1-1}x_3^{d_1+d_3-d_2+2j-2}x_2^{d_2-d_1-j+1}, \ldots, x_3^{d_1+d_3-d_2+2j-2}x_2^{d_2-j}\ for\ j = 1, \ldots, d_2 - d_1,$$

$$\{x_1, x_2\}^{d_1-j}x_3^{d_2+d_3-d_1-2+2j}\ for\ j = 1, \ldots, d_1).$$

Proof. We have $|J_{d_1}| = 1$, hence $J_{d_1} = \{x_1^{d_1}\}$, since J is strongly stable. Therefore:

$$Shad(J_{d_1}) = \{x_1^{d_1}\{x_1, x_2\}, x_1^{d_1}x_3\}.$$

Assume $d_2 > d_1 + 1$. Using the formulas from 2.3.17 we get $|J_{d_1+1}| - |Shad(J_{d_1})| = 0$, therefore

$$J_{d_1+1} = Shad(J_{d_1}) = \{x_1^{d_1}\{x_1, x_2\}, x_1^{d_1}x_3\}.$$

Using induction on $1 \leq j \leq d_2 - d_1 - 1$ we prove that

$$J_{d_1+j} = Shad(J_{d_1+j-1}) = \{x_1^{d_1}\{x_1, x_2\}^j, \ldots, x_3^jx_1^{d_1}\}.$$

Indeed, this assertion was already proved for $j = 1$, and if we suppose that is true for some $j < d_2 - d_1 - 1$ we get $|J_{d_1+j+1}| = |Shad(J_{d_1+j})|$ so we are done. In particular, we obtain $J_{d_2-1} = \{x_1^{d_1}\{x_1, x_2\}^{d_2-d_1-1}, x_3x_1^{d_1}\{x_1, x_2\}^{d_2-d_1-2}, \ldots, x_3^{d_2-d_1-1}x_1^{d_1}\}$.

We have $|J_{d_2}| - |Shad(J_{d_2-1})| = 1$ so we must add a new generator to $Shad(J_{d_2-1})$ in order to obtain J_{d_2}. But since J is strongly stable and x_3 is a strong Lefschetz element for S/J, this new generator is $x_1^{d_1-1}x_2^{d_2-d_1+1}$ and therefore $J_{d_2} = \{x_1^{d_1-1}\{x_1,x_2\}^{d_2-d_1+1}, x_3x_1^{d_1}\{x_1,x_2\}^{d_2-d_1-1}, \dots, x_3^{d_2-}$

We show by induction on $1 \le j \le d_3 - d_2$ that

$$J_{d_2-1+j} = Shad(J_{d_2-2+j}) \cup \{x_1^{d_1-j}x_2^{d_2-d_1+2j-1}\} =$$

$$= \{x_1^{d_1-j}\{x_1,x_2\}^{d_2-d_1+2j-1}, \dots, x_3^jx_1^{d_1}\{x_1,x_2\}^{d_2-d_1-1}, \dots, x_3^{d_2-d_1+j-1}x_1^{d_1}\}.$$

The first step of induction was already done. Suppose the assertion is true for some $j < d_3 - d_2$. Since $|J_{d_2+j}| - |Shad(J_{d_2-1+j})| = 1$ and J is strongly stable and x_3 is a strong Lefschetz element for S/J, we add $x_1^{d_1-j-1}x_2^{d_2-d_1+2j+1}$ to $Shad(J_{d_2-1+j})$ in order to obtain J_{d_2+j}. Thus, we are done. In particular, we get

$$J_{d_3-1} = \{x_1^{d_1+d_2-d_3}\{x_1,x_2\}^{2d_3-d_1-d_2-1}, x_3x_1^{d_1+d_2-d_3+1}\{x_1,x_2\}^{2d_3-d_1-d_2-3}, \dots,$$

$$x_3^{d_3-d_2}x_1^{d_1}\{x_1,x_2\}^{d_2-d_1-1}, \dots, x_3^{d_3-d_1-1}x_1^{d_1}\}.$$

We consider first $\alpha = 2$, i.e. $d_3 = d_1 + d_2 - 2$. In this case, since $|J_{d_3}| - |Shad(J_{d_3-1})| = 2$ we add two new generators to $Shad(J_{d_3-1})$ in order to obtain J_{d_3}. Since J is strongly stable and x_3 is a strong Lefschetz element for S/J, these new generators are $x_2^{d_3}$ and $x_1x_2^{d_3-1}$.

Suppose now $\alpha \ge 2$. We have $|J_{d_3}| - |Shad(J_{d_3-1})| = 2$ so we must add two new generators to $Shad(J_{d_3-1})$ to obtain J_{d_3}. Since J is strongly stable and x_3 is a strong Lefschetz element for S/J, these new generators are $x_1^{d_1+d_2-d_3-2}x_2^{2d_3-d_1-d_2+2}$ and $x_1^{d_1+d_2-d_3-1}x_2^{2d_3-d_1-d_2+1}$, therefore

$$J_{d_3} = \{x_1^{d_1+d_2-d_3-2}\{x_1,x_2\}^{2d_3-d_1-d_2+2}, x_3x_1^{d_1+d_2-d_3}\{x_1,x_2\}^{2d_3-d_1-d_2-1},$$

$$x_3^2x_1^{d_1+d_2-d_3+1}\{x_1,x_2\}^{2d_3-d_1-d_2-3}, \dots, x_3^{d_3-d_2}x_1^{d_1}\{x_1,x_2\}^{d_2-d_1}, \dots, x_3^{d_3-d_1}x_1^{d_1}\}.$$

One can prove by induction on $1 \le j \le \left\lfloor \frac{\alpha-1}{2} \right\rfloor$ that J_{d_3-1+j} is the set

$$Shad(J_{d_3-2+j}) \cup \{x_1^{d_1+d_2-d_3-2j}x_2^{2d_3-d_1-d_2+3j-1}, x_1^{d_1+d_2-d_3-2j+1}x_2^{2d_3-d_1-d_2+3j-2}\} =$$

$$\{x_1^{d_1+d_2-d_3-2j}\{x_1,x_2\}^{2d_3-d_1-d_2+3j-1}, \dots, x_1^{d_1+d_2-d_3-2}x_3^{j-1}\{x_1,x_2\}^{2d_3-d_1-d_2+2}, x_3^jJ_{d_3-1}\}.$$

Indeed, the assertion was proved for $j = 1$ and the induction step is similar. In the following, we must consider two possibilities: α is even or α is odd. Suppose first α is even. We obtain that $J_{\frac{d_1+d_2+d_3-4}{2}}$ is the set

$$\{x_1^2\{x_1,x_2\}^{\frac{d_1+d_2+d_3-8}{2}}, \dots, x_1^{d_1+d_2-d_3-2}x_3^{\frac{d_1+d_2-d_3-4}{2}}\{x_1,x_2\}^{2d_3-d_1-d_2+2}, x_3^{\frac{d_1+d_2-d_3-2}{2}}J_{d_3-1}\}.$$

We have $|J_{\frac{d_1+d_2+d_3-2}{2}}| - |Shad(J_{\frac{d_1+d_2+d_3-4}{2}})| = 2$ so we must add two generators to $Shad(J_{\frac{d_1+d_2+d_3-4}{2}})$ and, since J is strongly stable and x_3 is a strong Lefschetz element for S/J, these new generators are $x_2^{\frac{d_1+d_2+d_3-2}{2}}$ and $x_1x_2^{\frac{d_1+d_2+d_3-4}{2}}$. Therefore

$$J_{\frac{d_1+d_2+d_3-2}{2}} = \{\{x_1,x_2\}^{\frac{d_1+d_2+d_3-2}{2}}, x_1^2x_3\{x_1,x_2\}^{\frac{d_1+d_2+d_3-8}{2}}, \dots,$$

$$x_1^{d_1+d_2-d_3-2}x_3^{\frac{d_1+d_2-d_3-2}{2}}\{x_1,x_2\}^{2d_3-d_1-d_2+2}, x_3^{\frac{d_1+d_2-d_3}{2}}J_{d_3-1}\}.$$

One can easily prove by induction on $1 \leq j \leq \frac{\alpha-4}{2}$ that $J_{\frac{d_1+d_2+d_3-2}{2}+j}$ is the set

$$= Shad(J_{\frac{d_1+d_2+d_3-4}{2}+j}) \cup \{x_3^{2j}x_1^{2j-1}x_2^{\frac{d_1+d_2+d_3-2}{2}-3j}, \ldots, x_3^{2j}x_2^{\frac{d_1+d_2+d_3-2}{2}-j}\} =$$

$$\{\{x_1,x_2\}^{\frac{d_1+d_2+d_3-2}{2}+j}, \ldots, x_3^{2j}\{x_1,x_2\}^{\frac{d_1+d_2+d_3-2}{2}-j}, x_1^{2j+2}x_3^{2j+1}\{x_1,x_2\}^{\frac{d_1+d_2+d_3-8}{2}-3j}, \ldots,$$

$$x_1^{d_1+d_2-d_3-2}x_3^{\frac{d_1+d_2-d_3-2}{2}+j}\{x_1,x_2\}^{2d_3-d_1-d_2+2}, x_3^{\frac{d_1+d_2-d_3}{2}+j}J_{d_3-1}\}.$$

Suppose now α is odd. We have that $J_{\frac{d_1+d_2+d_3-3}{2}}$ is the set

$$\{x_1\{x_1,x_2\}^{\frac{d_1+d_2+d_3-5}{2}}, \ldots, x_1^{d_1+d_2-d_3-2}x_3^{\frac{d_1+d_2-d_3-3}{2}}\{x_1,x_2\}^{2d_3-d_1-d_2+2}, x_3^{\frac{d_1+d_2-d_3-1}{2}}J_{d_3-1}\}.$$

We have $|J_{\frac{d_1+d_2+d_3-1}{2}}| - |Shad(J_{\frac{d_1+d_2+d_3-3}{2}})| = 2$ so we must add two new generators to $Shad(J_{\frac{d_1+d_2+d_3-3}{2}})$ in order to obtain $J_{\frac{d_1+d_2+d_3-1}{2}}$. Since J is strongly stable and x_3 is a strong Lefschetz element for S/J, these new generators are $x_2^{\frac{d_1+d_2+d_3-1}{2}}$ and $x_3x_2^{\frac{d_1+d_2+d_3-3}{2}}$. Therefore

$$J_{\frac{d_1+d_2+d_3-1}{2}} = \{\{x_1,x_2\}^{\frac{d_1+d_2+d_3-1}{2}}, x_3\{x_1,x_2\}^{\frac{d_1+d_2+d_3-3}{2}}, x_1^3x_3^2\{x_1,x_2\}^{\frac{d_1+d_2+d_3-11}{2}}, \ldots,$$

$$x_1^{d_1+d_2-d_3-2}x_3^{\frac{d_1+d_2-d_3-1}{2}}\{x_1,x_2\}^{2d_3-d_1-d_2+2}, x_3^{\frac{d_1+d_2-d_3+1}{2}}J_{d_3-1}\}.$$

One can easily prove by induction on $1 \leq j \leq \frac{\alpha-3}{2}$ that

$$J_{\frac{d_1+d_2+d_3-1}{2}+j} = Shad(J_{\frac{d_1+d_2+d_3-3}{2}+j}) \cup \{x_1^{2j}x_3^{2j+1}x_2^{\frac{d_1+d_2+d_3-3}{2}-3j}, \ldots, x_3^{2j+1}x_2^{\frac{d_1+d_2+d_3-3}{2}-j}\}.$$

For $j=1$, we notice that $|J_{\frac{d_1+d_2+d_3+1}{2}}| - |Shad(J_{\frac{d_1+d_2+d_3-1}{2}})| = 3$ so we must add 3 new monomials to $Shad(J_{\frac{d_1+d_2+d_3-1}{2}})$ in order to obtain $J_{\frac{d_1+d_2+d_3+1}{2}}$. But, since J is strongly stable and x_3 is a strong Lefschetz element for S/J, they are exactly $x_1^2x_3^3x_2^{\frac{d_1+d_2+d_3-3}{2}-3}$, $x_1x_3^3x_2^{\frac{d_1+d_2+d_3-3}{2}-2}$ and $x_3^3x_2^{\frac{d_1+d_2+d_3-3}{2}-1}$, as required. The induction step is similar.

In all cases, we obtain that $J_{d_1+d_2-2}$ is the set

$$\{\{x_1,x_2\}^{d_1+d_2-2}, x_3\{x_1,x_2\}^{d_1+d_2-3}, \ldots, x_3^{d_1+d_2-d_3-2}\{x_1,x_2\}^{d_3}, x_3^{d_1+d_2-d_3-1}J_{d_3-1}\}.$$

We have $|J_{d_1+d_2-1}| - |Shad(J_{d_1+d_2-2})| = \alpha$, so we must add α new generators to obtain $J_{d_1+d_2-1}$. Since J is strongly stable and x_3 is a strong Lefschetz element for S/J, they are $x_1^{d_1+d_2-d_3-1}x_2^{2d_3-d_1-d_2}x_3^{d_1+d_2-d_3}, \ldots, x_2^{d_3-1}x_3^{d_1+d_2-d_3}$, therefore $J_{d_1+d_2-1}$ is the set

$$\{\{x_1,x_2\}^{d_1+d_2-1}, \ldots, x_3^{d_1+d_2-d_3}\{x_1,x_2\}^{d_3-1}, x_3^{d_1+d_2-d_3+1}x_1^{d_1+d_2-d_3+1}\{x_1,x_2\}^{2d_3-d_1-d_2-3},$$

$$\ldots, x_3^{d_1}x_1^{d_1}\{x_1,x_2\}^{d_2-d_1-1}, x_1^{d_1}x_3^{d_1+1}\{x_1,x_2\}^{d_2-d_1-2}, \ldots, x_1^{d_1}x_3^{d_2-1}\}.$$

One can easily prove by induction on $1 \leq j \leq d_3 - d_2$ that $J_{d_1+d_2-1+j}$ is the set

$$Shad(J_{d_1+d_2-2+j}) \cup \{x_1^{d_1+d_2-d_3+j-1} x_2^{2d_3-d_1-d_2-2j} x_3^{d_1+d_2-d_3+2j}, \dots, x_2^{d_3-1-j} x_3^{d_1+d_2-d_3+2j}\}$$

Indeed, the case $j = 1$ was already done and the induction step is similar. In particular, we get

$$J_{d_1+d_3-2} = \{\{x_1, x_2\}^{d_1+d_3-2}, x_3\{x_1, x_2\}^{d_1+d_3-3}, \dots, x_3^{d_1+d_3-d_2-2}\{x_1, x_2\}^{d_2},$$
$$x_1^{d_1} x_3^{d_1+d_3-d_2-1}\{x_1, x_2\}^{d_2-d_1-1}, \dots, x_1^{d_1} x_3^{d_3-2}\}.$$

Since $|J_{d_1+d_3-1}| - |Shad(J_{d_1+d_3-2})| = d_1$ we must add d_1 generators to $Shad(J_{d_1+d_3-2})$ in order to obtain $J_{d_1+d_3-1}$. Since J is strongly stable and x_3 is a strong Lefschetz element for S/J, these new generators are $x_1^{d_1-1} x_3^{d_1+d_3-d_2} x_2^{d_2-d_1}, \dots, x_3^{d_1+d_3-d_2} x_2^{d_2-1}$,so $J_{d_1+d_3-1}$ is the set

$$\{\{x_1, x_2\}^{d_1+d_3-1}, ..., x_3^{d_1+d_3-d_2}\{x_1, x_2\}^{d_2-1}, x_1^{d_1} x_3^{d_1+d_3-d_2+1}\{x_1, x_2\}^{d_2-d_1-2}, ..., x_1^{d_1} x_3^{d_3-1}\}.$$

We prove by induction on $1 \leq j \leq d_2 - d_1$ that $J_{d_1+d_3-2+j}$ is the set

$$Shad(J_{d_1+d_3-3+j}) \cup \{x_1^{d_1-1} x_3^{d_1+d_3-d_2+2j-2} x_2^{d_2-d_1-j+1}, \dots, x_3^{d_1+d_3-d_2+2j-2} x_2^{d_2-j}\}.$$

Indeed, we already proved this for $j = 1$ and the induction step is similar. We get

$$J_{d_2+d_3-2} = \{\{x_1, x_2\}^{d_1+d_3-2}, x_3\{x_1, x_2\}^{d_1+d_3-3}, \dots, x_3^{d_3+d_2-d_1-2}\{x_1, x_2\}^{d_1}\}.$$

One can easily prove by induction on $1 \leq j \leq d_1$ that

$$J_{d_2+d_3-2+j} = Shad(J_{d_2+d_3-3+j}) \cup \{x_3^{d_2+d_3-d_1-2+2j}\{x_1, x_2\}^{d_1-j}\} =$$
$$= \{\{x_1, x_2\}^{d_2+d_3-2+j}, \dots, x_3^{d_2+d_3-d_1-2+2j}\{x_1, x_2\}^{d_1-j}\}.$$

Finally, we obtain $J_{d_1+d_2+d_3-2} = S_{d_1+d_2+d_3-2}$ and therefore we cannot add new minimal generators of J in degrees $> d_1 + d_2 + d_3 - 2$. □

Corollary 2.3.19. *In the above conditions of the above proposition, the number of minimal generators of J is $d_1(d_2 + 1) - \left(\frac{\alpha}{2}\right)^2 + 1$ if α is even or $d_1(d_2 + 1) - \frac{\alpha^2-1}{4} + 1$ if α is odd.*

Example 2.3.20. *1. Let $d_1 = 3$, $d_2 = 5$ and $d_3 = 6$. Then*

$$J = (x_1^3, x_1^2 x_2^3, x_1 x_2^5, x_2^6,\ x_2^4 x_3^2\{x_1, x_2\},\ x_2^2 x_3^4\{x_1, x_2\}^2,$$
$$x_2 x_3^6\{x_1, x_2\}^2,\ x_3^8\{x_1, x_2\}^2,\ x_3^{10}\{x_1, x_2\},\ x_3^{12}).$$

2. Let $d_1 = 4$, $d_2 = 5$ and $d_3 = 6$. Then

$$J = (x_1^4,\ x_1^3 x_2^2,\ x_1^2 x_2^4,\ x_1 x_2^5,\ x_2^7,\ x_3 x_2^6,\ x_2^3 x_3^3\{x_1, x_2\}^2,$$
$$x_2 x_3^5\{x_1, x_2\}^3,\ x_3^7\{x_1, x_2\}^3,\ x_3^9\{x_1, x_2\}^2,\ x_3^{11}\{x_1, x_2\},\ x_3^{13}).$$

Remark 2.3.21. *If $f_1, f_2, f_3 \in S = K[x_1, x_2, x_3]$ is a regular sequence of homogeneous polynomials of given degrees d_1, d_2, d_3 such that $S/(f_1, f_2, f_3)$ has (SLP), then the number of minimal generators of $J = Gin((f_1, f_2, f_3))$, $\mu(J) \leq d_1(d_2+1)+1$. This follows immediately from* 2.2.4*,* 2.2.9*,* 2.3.4*,* 2.3.9*,* 2.3.14 *and* 2.3.19*.*

Remark 2.3.22. *Let $f_1, f_2, f_3 \in S = K[x_1, x_2, x_3]$ be a regular sequence of homogeneous polynomials of given degrees d_1, d_2, d_3 such that S/I has (SLP), where $I = (f_1, f_2, f_3)$. Let $J = Gin(I)$. One can compute the socle of S/J, as follows. Since J is (strongly) stable, $(J : (x_1, x_2, x_3)) = (J : x_3)$. Indeed, if $u \in (J : x_3)$, then $x_3 u \in J$ and since J is stable, $x_1(x_3 u)/x_3 = x_1 u \in J$ and also $x_2(x_3 u)/x_3 = x_2 u \in J$, thus $u \in (J : (x_1, x_2, x_3))$.*

On the other hand, for example, when $d_1 + d_2 \leq d_3 + 1$ and $d_1 = d_2 < d_3$, Proposition 2.2.3 *implies $(J : x_3) = J + T$, where*

$$T = (x_2^{2d-2j-2} x_3^{d_3-2d+2j+1}\{x_1, x_2\}^j,\ 0 \leq j \leq d-2,\ x_3^{d_3+2j-3}\{x_1, x_2\}^{d-j},\ 1 \leq j \leq d).$$

One can check that none of the minimal generators of T is in J. Therefore, the set of the minimal generators of T form a base for $Soc(S/J)$.

The other cases are similar, and the reader can easily compute the socle of S/J for any integers $d_1, d_2, d_3 \geq 2$.

2.4 Generic initial ideal for (n, d)-complete intersections.

Let K be an algebraically closed field of characteristic zero. Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over K. Let $n, d \geq 2$ be two integers. We consider $I = (f_1, \ldots, f_n) \subset S$ an ideal generated by a regular sequence $f_1, \ldots, f_n \in S$ of homogeneous polynomials of degree d. We say that $A = S/I$ is a (n, d)-*complete intersection*. Let $J = Gin(I)$ be the generic initial of I, with respect to the reverse lexicographic (revlex) order (see [18, §15.9], for details).

We say that a property (P) holds for a generic sequence of homogeneous polynomials $f_1, f_2, \ldots, f_n \in S$ of given degrees $d_1, d_2, \ldots, d_n$ if there exists a nonempty open Zariski subset $U \subset S_{d_1} \times S_{d_2} \times \cdots \times S_{d_n}$ such that for every n-tuple $(f_1, f_2, \ldots, f_n) \in U$ the property (P) holds.

For any nonnegative integer k, we denote by J_k the set of monomials of J of degree k. Conca and Sidman proved that J_d is revlex if $f_1, \ldots, f_n$ is a generic regular sequence, (see [16, Theorem 1.2]). We will prove that J_d is a revlex set in another case, namely, when $f_i \in k[x_i, \ldots, x_n]$. It is likely to be true that J_d is revlex for any complete intersection, but we do not have the means to prove this assertion.

Let $I = (f_1, \ldots, f_n) \subset S = K[x_1, \ldots, x_n]$ be an ideal generated by a regular sequence $f_1, \ldots, f_n \in S$ of homogeneous polynomials of degree d. Let $J = Gin(I)$ be the generic initial ideal of I, with respect to the revlex order. It is well known that the Hilbert series of S/J is the same as the Hilbert series of S/I and moreover, $H(S/J, t) = H(S/I, t) = (1 + t + \cdots + t^{d-1})^n$. More precisely, we have:

Proposition 2.4.1. *1. $H(S/J, k) = \binom{k+n-1}{n-1}$, for $0 \le k \le d-1$.*

2. $H(S/J, k) = \binom{k+n-1}{n-1} - n\binom{j+n-1}{n-1}$, for $d \le k \le \left\lfloor \frac{n(d-1)}{2} \right\rfloor$ and $j = k - d$.

3. $H(S/J, k) = H(S/J, n(d-1) - k)$, for $k \ge \left\lceil \frac{n(d-1)}{2} \right\rceil$.

Proof. Use induction on n. Denote $H_n(t) = (1 + t + \cdots + t^{d-1})^n$. The case $n = 1$ is trivial. The induction step follows from the equality $H_n(t) = H_{n-1}(t)(1 + t + \cdots + t^{d-1})$. □

Corollary 2.4.2. *1. $|J_k| = 0$, for $k \le d-1$.*

2. $|J_k| = n\binom{j+n-1}{n-1}$, for $d \le k \le \left\lfloor \frac{n(d-1)}{2} \right\rfloor$ and $j = k - d$.

3. $|J_k| = \binom{\lceil \frac{n(d-1)}{2} \rceil + j + n - 1}{n-1} - \binom{\lfloor \frac{n(d-1)}{2} \rfloor - j + n - 1}{n-1} + n\binom{\lfloor \frac{n(d-1)}{2} \rfloor - d - j - n}{n-1}$, for $\left\lceil \frac{n(d-1)}{2} \right\rceil \le k \le (n-1)(d-1) - 1$, where $j = k - \left\lceil \frac{n(d-1)}{2} \right\rceil$

4. $|J_k| = \binom{(n-1)d+j}{n-1} - \binom{n-1+d-1-j}{n-1}$, for $(n-1)(d-1) \le k \le n(d-1)$, where $j = k - (n-1)(d-1)$.

Proof. Using $|J_k| = |S_k| - H(S/J, k)$ the proof follows from 2.4.1. □

Suppose $f_i = \sum_{k=1}^{N} b_{ik} u_k$ for $1 \le i \le n$ where $u_1, u_2, \ldots, u_N \in S$ are all the monomials of degree d decreasing ordered in revlex and $N = \binom{d+n-1}{n-1}$. We denote $u_k = x^{\alpha_k}$. For example, $\alpha_1 = (d, 0, \ldots, 0)$, $\alpha_2 = (d-1, 1, 0, \ldots, 0)$ etc.

We take a generic transformation of coordinates $x_i \mapsto \sum_{j=1}^{n} c_{ij} x_j$ for $i = 1, \ldots, n$. Conca and Sidman proved in [6] that we may assume that c_{ij} are algebraically independents over K. More precisely, if we consider the field extension $K \subset L = K(c_{ij} | i, j = \overline{1, n})$ and if we set

$$F_i = f_i\left(\sum_{j=1}^{n} c_{1j} x_j, \ldots, \sum_{j=1}^{n} c_{nj} x_j\right) \in L[x_1, \ldots, x_n],\ i = 1, \ldots, n$$

then $J = Gin(I) = in(F_1, \ldots, F_n) \cap S$.

We write $F_i = \sum_{j=1}^{n} a_{ij} u_j + \cdots$ the monomial decomposition of F_i in $L[x_1, \ldots, x_n]$. With these notations, we have the following elementary lemma:

Lemma 2.4.3. *J_d is revlex if and only if the following condition is fulfilled:*

$$\Delta = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} \ne 0.$$

Proof. Suppose $\Delta \neq 0$. Since $|J_d| = n$, it is enough to show that $u_1, \dots, u_n \in J$. Let $A = (a_{ij})_{\substack{i=1,n \\ j=1,n}}$. Since $\Delta = det(A) \neq 0$, A is invertible and we have

$$A^{-1}\begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix} = \begin{pmatrix} H_1 \\ \vdots \\ H_n \end{pmatrix},$$

where $H_i = u_i+$ small terms in revlex order. Therefore $LM(H_i) = u_i \in J$, for all $1 \leq i \leq n$, where $LM(H_i)$ denotes the leading monomial of H_i in the revlex order.

Conversely, since $u_1, \dots, u_n \in J_d$, we can find some polynomials $H_i \in L[x_1, \dots, x_n]$, with $LM(H_i) = u_i$, $1 \leq i \leq n$, as linear combination of F_i's. If we denote $H_i = \sum_{j=1}^{N} \widetilde{a}_{ij} u_j$ and $\widetilde{A} = (\widetilde{a}_{ij})_{i,j=1,\dots,n}$, it follows that there exists a map $\psi : L^n \to L^n$, given by a matrix $E = (e_{ij})_{i,j=1,\dots,n}$, such that $\widetilde{A} = A \cdot E$. Now, since $det(\widetilde{A}) \neq 0$ it follows that $\Delta = det(A) \neq 0$, as required. $\square$

Remark 2.4.4. By the changing of variables φ given by $x_i \mapsto \sum_{j=1}^{n} c_{ij} x_j$, x^{α_k} became

$$m_k := (\sum_{j=1}^{n} c_{1j} x_j)^{\alpha_{k1}} \cdots (\sum_{j=1}^{n} c_{nj} x_j)^{\alpha_{kn}} = (\sum_{|t|=\alpha_{k1}} c_1^t x^t) \cdots (\sum_{|t|=\alpha_{kn}} c_n^t x^t),$$

where, for any multiindex $t = (t_1, \dots, t_n)$ we denoted $x^t = x_1^{t_1} \cdots x_n^{t_n}$ and $c_i^t = c_{i1}^{t_1} \cdots c_{in}^{t_n}$. Let g_{kl} be the coefficient in c_{ij}'s of x^{α_l} in the monomial decomposition of m_k. Using the above writing of m_k, we claim that:

$$(1)\quad g_{kl} = \sum_{\substack{|t_1| = \alpha_{k1}, \dots, |t_n| = \alpha_{kn} \\ t_1 + \cdots + t_n = \alpha_l}} \left[\binom{\alpha_{k1}}{t_{11}} \cdots \binom{\alpha_{kn}}{t_{n1}}\right] \left[\binom{\alpha_{k1} - t_{11}}{t_{12}} \cdots \binom{\alpha_{kn} - t_{n1}}{t_{n2}}\right] \cdots$$

$$\left[\binom{\alpha_{k1} - t_{11} - \cdots t_{1n-1}}{t_{1n}} \cdots \binom{\alpha_{kn} - t_{n1} - \cdots - t_{nn-1}}{t_{nn}}\right] \cdot c_1^{t_1} \cdots c_n^{t_n}.$$

Indeed, the monomial $c_1^{t_1} \cdots c_n^{t_n}$ appear in the coefficient of x^{α_l} in the expansion of m_k if and only if $t_1 + \cdots + t_n = \alpha_l$ and $|t_1| = \alpha_{k1}, \dots, |t_n| = \alpha_{kn}$. Moreover, by Newton binomial, the coefficient of $x_1^{t_{i1}} \cdots x_n^{t_{in}}$ in $(\sum_{j=1}^{n} c_{ij} x_j)^{\alpha_{k1}}$ is $\binom{\alpha_{k1}}{t_{i1}}\binom{\alpha_{k1}-t_{i1}}{t_{i2}} \cdots \binom{\alpha_{k1}-t_{i1}-\cdots t_{i,n-1}}{t_{in}} c_i^{t_i}$ for any $1 \leq i \leq n$, and thus we proved the claim.

Since $a_{il} = \sum_{k=1}^{N} b_{ik} \cdot g_{kl}$, from the Cauchy-Binet formula we get:

$$\Delta = \sum_{1 \leq k_1 < k_2 < \ldots < k_n \leq N} B_{k_1,k_2,\dots,k_n} G_{k_1,k_2,\dots,k_n}, \ \textit{where}$$

$$B_{k_1,k_2,\dots,k_n} = \begin{vmatrix} b_{1k_1} & \cdots & b_{1k_n} \\ \vdots & & \vdots \\ b_{nk_1} & \cdots & b_{nk_n} \end{vmatrix} \ \textit{and} \ G_{k_1,k_2,\dots,k_n} = \begin{vmatrix} g_{k_1 1} & \cdots & g_{k_n 1} \\ \vdots & & \vdots \\ g_{k_1 n} & \cdots & g_{k_n n} \end{vmatrix}.$$

Now, we are able to prove the main result of our paper.

Theorem 2.4.5. *If $f_i \in K[x_i, \dots, x_n]$ then J_d is revlex. In particular, if S/I is a monomial complete intersection, then J_d is revlex.*

Proof. Let $k_i = \binom{i+d-1}{d}$, for any $i = 1, \dots, n$. Then $u_{k_i} = x_i^d$. Recall our notation, $u_k = x^{\alpha_k}$. We have $b_{11} \neq 0$, otherwise $I = (f_1, \dots, f_n) \subset (x_2, \dots, x_n)$ contradicting the fact that I is an Artinian ideal. Using a similar argument, we get $b_{ik_i} \neq 0$ for all $1 \leq i \leq n$. Thus, multiplying each f_i with $b_{ik_i}^{-1}$, we may assume $b_{ik_i} = 1$ for all $1 \leq i \leq n$. In other words, $f_i = x_i^d + f_i'$, where f_i' contains monomials smaller than x_i^d in the revlex order. Also, since $f_i \in K[x_i, \dots, x_n]$ we have $b_{i'k_i} = 0$ for any $i' > i$. In particular,$B_{k_1,\dots,k_n} = 1$.

In the expansion of the determinant $G_{k_1,\dots,k_n}$, appears the term $g_{k_1 1} \cdot g_{k_2 2} \cdots g_{k_n n} = r \cdot (c_{11}^d)(c_{21}^{d-1}c_{22}) \cdots (c_i^{\alpha_i}) \cdots (c_n^{\alpha_n})$, where r is a nonzero (positive) integer. Indeed, by (1), we have $g_{11} = c_{11}^d$, $g_{k_2 2} = d c_{21}^{d-1} c_{22}$ and, in general, $g_{k_i i} =$ some binomial coefficient $\cdot c_i^{\alpha_i}$. We claim that $m = (c_{11}^d)(c_{21}^{d-1}c_{22}) \cdots (c_i^{\alpha_i}) \cdots (c_n^{\alpha_n})$ doesn't appear again in the expansion of Δ.

Since $f_i \in k[x_i, \dots, x_n]$, in the monomials in (c_{tl}) of a_{ij} there are no c_{tl}'s with $t < i$. Also, all the monomials of f_i' contain variables x_t with $t > i$. Corresponding to them, in a_{ij}'s there are c_{tj}'s with $t > i$. Thus in a_{il} the only monomials in $c_{i1}, \dots, c_{in}$ of degree d comes from $\varphi(x_i^d) = (\sum_{j=1}^n c_{ij}x_j)^d$, the other monomials being multiples of some c_{tl} with $t > i$. Consequently, in the expansion of Δ, the monomials of the type $c_1^{\beta_1} \cdots c_n^{\beta_n}$, where $\beta_1, \dots, \beta_n$ are multiindices with $|\beta_1| = \cdots = |\beta_n| = d$ comes only from $\varphi(x_1^d), \dots, \varphi(x_n^d)$.

On the other hand, for any $1 \leq i \leq n$, $c_i^{\alpha_i}$ is unique between the monomials in c_{tl}'s from $\varphi(x_n^d)$, because they are of the type c_i^{γ}, where γ is a multiindex with $|\gamma| = d$. From these facts, it follows that the monomial m is unique in the monomial expansion of Δ and occurs there with a nonzero coefficient. Thus $\Delta \neq 0$ and by applying Lemma 2.4.3 we complete the proof of the theorem. $\square$

Remark 2.4.6. In the case $n = 2$ and $n = 3$, J_d is revlex for any (n, d)-complete intersection. Indeed, in the case $n = 2$, J itself is revlex since it is strongly stable. In the case $n = 3$, since $|J_d| = 3$ and J is strongly stable, it follows that either (a) $J_d = (x_1^d, x_1^{d-1}x_2, x_1^{d-2}x_2^2)$, either (b) $J_d = (x_1^d, x_1^{d-1}x_2, x_1^{d-1}x_3)$. But in the case (b), the map $(S/J)_{d-1} \xrightarrow{\cdot x_3} (S/J)_d$ is not injective, because $x_1^{d-1} \neq 0$ in $(S/J)_{d-1}$ and $x_1^{d-1}x_3 = 0$ in $(S/J)_d$. This is a contradiction with the fact that x_3 is a weak Lefschetz element on S/J and therefore, J_d is revlex.

Lemma 2.4.7. *(a) $a_{i1} = f_i(c_{11}, \dots, c_{n1})$ for all $1 \leq i \leq n$.*

(b) If $1 \leq l \leq n$ is an integer then the sequence $a_{1l}, a_{2l}, \dots, a_{nl}$ is regular as a sequence of polynomials in $K[c_{ij}|\ 1 \leq i, j \leq n]$.

Proof. Substituting $x_j = 0$ for $j \neq 1$ in F_i we get (a). In order to prove (b), firstly notice that $a_{11}, a_{21}, \dots, a_{n1}$ is a regular sequence on $K[c_{11}, c_{21}, \dots, c_{n1}]$, since $f_1, \dots, f_n$ is a regular sequence on $K[x_1, \dots, x_n]$ and $c_{11}, c_{21}, \dots, c_{n1}$ are algebraically independent over K.

Let $1 \leq l \leq n$ be an integer. We claim that

$$(*)\ \frac{K[c_{ij}|\ 1 \leq i, j \leq n]}{(a_{1l}, \dots, a_{nl}, c_{i1} - c_{ij}\ ,\ 1 \leq i \leq n,\ 2 \leq j \leq n)} \cong \frac{K[c_{11}, c_{21}, \dots, c_{n1}]}{(a_{11}, a_{21}, \dots, a_{n1})}.$$

Indeed, by (1), if we put $c_{ij} = c_{i1}$ for all $1 \leq i \leq n$ and $2 \leq j \leq n$ in the expansion of g_{kl} we obtain $r_l \cdot g_{k1}$, where r_l is a strictly positive integer, which depends only on l, and therefore, a_{il} became $r_l \cdot a_{i1}$. From (*) it follows that $a_{1l}, \ldots, a_{nl}, c_{i1} - c_{ij}\ for\ 1 \leq i \leq n,\ 2 \leq j \leq n$ is a system of parameters for $K[c_{ij}|\ 1 \leq i,j \leq n]$ and thus $a_{1l}, \ldots, a_{nl}$ is a regular sequence on $K[c_{ij}|\ 1 \leq i,j \leq n]$, so we proved (b). $\square$

As we noticed in Remark 2.4.6, for $n = 3$, the conclusion of Theorem 2.4.5 holds for any regular sequence f_1, f_2, f_3 of homogeneous polynomials of degree d. In the following, we give another proof of this, without using the fact that $S/(f_1, f_2, f_3)$ has the (WLP), i.e. x_3 is a weak Lefschetz element for S/J. Also, we get the same conclusion for the case $n = 4$ and $d = 2$. However, this approach do not works in the general case.

Proposition 2.4.8. *(a) If $f_1, f_2, f_3 \in K[x_1, x_2, x_3]$ is a regular sequence of homogeneous polynomials of degree $d \geq 2$, $I = (f_1, f_2, f_3)$ and $J = Gin(I)$, the generic initial ideal of I, with respect to the reverse lexicographical order, then J_d is a revlex set.*

(b) If $f_1, f_2, f_3, f_4 \in K[x_1, x_2, x_3, x_4]$ is a regular sequence of homogeneous polynomials of degree 2, $I = (f_1, f_2, f_3, f_4)$ and $J = Gin(I)$, the generic initial ideal of I, with respect to the reverse lexicographical order, then J_2 is a revlex set.

Proof. (a) Let $A = (a_{ij})_{i,j=\overline{1,3}}$. Since $Gin(f_1, f_2)$ is strongly stable, it follows by Lemma 2.4.3 that $\Delta_3 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$. Analogously, $\Delta_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \neq 0$ and $\Delta_1 = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \neq 0$. We have $\Delta = a_{13}\Delta_1 - a_{23}\Delta_2 + a_{33}\Delta_3$. Suppose $\Delta = 0$. It follows $a_{13}\Delta_1 = a_{23}\Delta_2 - a_{33}\Delta_3$ and therefore, since a_{13}, a_{23}, a_{33} is a regular sequence in $K[c_{ij}|i,j = \overline{1,3}]$, we get $\Delta_1 \in (a_{23}, a_{33})$. The first three monomials of degree d in revlex order are x_1^d, $x_1^{d-1}x_2$ and $x_1^{d-2}x_2^2$. It follows that the degree of a_{i1}, a_{i2} and a_{i3} in c_{21}, c_{22}, c_{23} is 0, 1, respectively 2, for any $1 \leq i \leq 3$. Therefore, the degree of Δ_1 in the variables c_{21}, c_{22}, c_{23} is 1, but the degree of a_{23} and a_{33} in c_{21}, c_{22}, c_{23} is 2, which is impossible, since $\Delta_1 \in (a_{23}, a_{33})$.

(b) Let $A = (a_{ij})_{i,j=\overline{1,4}}$. Since any three polynomials from f_1, f_2, f_3, f_4 form a regular sequence, it follows from (a) that any 3×3 minor of the matrix $\widetilde{A} = (a_{ij})_{\substack{i = \overline{1,4} \\ j = \overline{1,3}}}$ is nonzero. Let Δ_i be the minor obtained from $\widetilde{A}$ by erasing the i-row. Suppose $\Delta = 0$. It follows that $a_{14}\Delta_1 = a_{24}\Delta_2 - a_{34}\Delta_3 + a_{44}\Delta_4$ and therefore, since $a_{14}, a_{24}, a_{34}, a_{44}$ is a regular sequence in $K[c_{ij}|i,j = \overline{1,4}]$, we get $\Delta_1 \in (a_{24}, a_{34}, a_{44})$. Since the first 4 monomials in revlex are $x_1^2, x_1x_2, x_2^2, x_1x_3$, we get a contradiction from the fact that the degree of Δ_1 in the variables $c_{31}, c_{32}, c_{33}, c_{34}$ is zero, but the degree of a_{24}, a_{34}, a_{44} in $c_{31}, c_{32}, c_{33}, c_{34}$ is 1. $\square$

Remark 2.4.9. The hypothesis that K is a field with $char(K) = 0$ is essential. Indeed, suppose $char(K) = p$ and $I = (x_1^p, x_2^p) \subset K[x_1, x_2]$. Then, simply using the definition of the generic initial ideal, we get $Gin(I) = I$ and, obviously, $I_p = \{x_1^p, x_2^p\}$ is not revlex.

Also, the hypothesis that $f_1, \ldots, f_n$ is a regular sequence of homogeneous polynomials is essential. Let $I = (f_1, f_2, f_3) \subset K[x_1, x_2, x_3]$, where $f_1 = x_1^2$, $f_2 = x_1x_2$ and $f_3 = x_1x_3$. In order to compute the generic initial ideal of I we can take a generic transformation of coordinates with an upper triangular matrix, i.e. $x_1 \mapsto x_1,\ x_2 \mapsto x_2 + c_{12}x_1,\ x_3 \mapsto$

$x_3 + c_{23}x_2 + c_{13}x_1$, where $c_{ij} \in K$ for all i, j (see [18, §15.9]). We get

$$F_1(x_1, x_2, x_3) := f_1(x_1, x_2 + c_{12}x_1, x_3 + c_{23}x_2 + c_{13}x_1) = x_1^2,$$

$$F_2(x_1, x_2, x_3) := f_2(x_1, x_2 + c_{12}x_1, x_3 + c_{23}x_2 + c_{13}x_1) = c_{12}x_1^2 + x_1x_2,$$

$$F_3(x_1, x_2, x_3) := f_3(x_1, x_2 + c_{12}x_1, x_3 + c_{23}x_2 + c_{13}x_1) = c_{13}x_1^2 + c_{23}x_1x_2 + x_1x_3.$$

The generic initial ideal of I, $J = in(F_1, F_2, F_3)$ satisfies $J_2 = I_2$, but I_2 is not revlex.

2.5 Several examples of computation of the Gin.

Let $I = (f_1, \dots, f_n) \subset S = K[x_1, \dots, x_n]$ be an ideal generated by a regular sequence $f_1, \dots, f_n \in S$ of homogeneous polynomials of degree d. Let $J = Gin(I)$ be the generic initial ideal of I, with respect to the revlex order.

In section 2.3, the case $n = 3$ and $d \geq 2$ was treated completely, when $S/(f_1, f_2, f_3)$ has (SLP), see Proposition 2.3.3. In the following, we discuss some particular cases with $n \geq 4$.

The case $n = 4$, $d = 2$. We assume that S/I has (SLP). From Wiebe's Theorem, it follows that x_4 is a strong Lefschetz element for S/J. For a positive integer k, we denote $Shad(J_k) = \{x_1, \dots, x_n\}J_k$. We have $H(S/J, t) = (1 + t)^4 = 1 + 4t + 6t^2 + 4t^3 + t^4$.

We have $|J_2| = 4$. From Proposition 2.4.8, J_2 is revlex, therefore

$$J_2 = \{x_1^2, x_1x_2, x_2^2, x_1x_3\} = \{\{x_1, x_2\}^2, x_1x_3\}.$$

We have $|Shad(J_2)| = 12$. On the other hand, $|J_3| = 16$, so we need to add 4 new generators at $Shad(J_2)$ to get J_3. If we add a new monomial which is divisible by x_4^2, then the map $(S/J)_1 \xrightarrow{\cdot x_4^2} (S/J)_3$, will be no longer injective. Since $|(S/J)_1| = |(S/J)_3|$, we get a contradiction with the fact that x_4 is a strong Lefschetz element for S/J. But there exists only 16 monomials in S which are not multiple of x_4^2. Thus

$$J_3 = \{\{x_1, x_2, x_3\}^3, x_4\{x_1, x_2, x_3\}^2\}, \textit{ and therefore}$$

$$Shad(J_3) = \{\{x_1, x_2, x_3\}^4, x_4\{x_1, x_2, x_3\}^3, x_4^2\{x_1, x_2, x_3\}^2\}.$$

Since $|Shad(J_3)| = 31$ and $|J_4| = |S_4| - |(S/J)_4| = 35 - 1 = 34$ we have to add 3 new generators at $Shad(J_3)$ in order to get J_4. Since J is strongly stable, these new generators are $x_4^3x_1$, $x_4^3x_2$ and $x_4^3x_3$. So

$$J_4 = \{x_1, x_2, x_3, x_4\}^4 \setminus \{x_4^4\}. \textit{ We get } \; Shad(J_4) = \{x_1, x_2, x_3, x_4\}^5 \setminus \{x_4^5\}$$

and since $J_5 = S_5$ it follows that we must add x_4^5 at $Shad(J_4)$ to obtain J_5. From now one, we cannot add any new monomial. J is the ideal generated by all monomials added at some step k to $Shad(J_k)$, thus we proved the following proposition:

Proposition 2.5.1. *If $I = (f_1, f_2, f_3, f_4)$ is an ideal generated by a regular sequence of homogeneous polynomials $f_1, f_2, f_3, f_4 \in S = k[x_1, x_2, x_3, x_4]$ of degree 2 such that the algebra S/I has (SLP) then the generic initial ideal of I with respect to the revlex order is*

$$J = (x_1^2,\ x_1x_2,\ x_2^2,\ x_1x_3,\ x_2x_3^3,\ x_3^3,\ x_3^2x_4,\ x_3^2x_4,\ x_4^3x_1,\ x_4^3x_2,\ x_4^3x_3,\ x_4^5).$$

In particular, this assertion holds for a generic sequence of homogeneous polynomials $f_1, f_2, f_3, f_4 \in S$ or if $f_i \in k[x_i, \dots, x_4]$, $1 \le i \le 4$.

The case $n = 5$, $d = 2$. In the following, we suppose that S/I has (SLP), so x_5 is a strong Lefschetz element for S/J. Also, we suppose that J_2 is revlex. We have $H(S/J, t) = (1+t)^5 = 1 + 5t + 10t^2 + 10t^3 + 5t^4 + t^5$. We have $|J_2| = 5$. Since J_2 is revlex from the assumption, we have $J_2 = \{\{x_1, x_2\}^2, x_3\{x_1, x_2\}\}$. So

$$Shad(J_2) = \{\{x_1, x_2\}^3, \{x_1, x_2\}^2\{x_3, x_4, x_5\}, x_3\{x_1, x_2\}\{x_3, x_4, x_5\}\}.$$

We have $|Shad(J_2)| = 19$. On the other hand $|J_3| = |S_3| - |(S/J)_3| = 35 - 10 = 25$, so we must add 6 new generators, from a list of 16 monomials, at $Shad(J_2)$ to get J_3.

Since x_5 is a strong Lefschetz element for S/J it follows that we cannot add any monomial of the form $x_5 \cdot m$, where m is nonzero in $(S/J)_2$ because, in that case, the map $(S/J)_2 \overset{\cdot x_5}{\to} (S/J)_3$ will be no longer injective. But there are $|(S/J)_2| = 10$ such monomials m. Therefore, we must add the remaining 6 monomials, $x_3^3, x_3^2x_4, x_1x_4^2, x_2x_4^2, x_3x_4^2, x_4^3$. Thus

$$J_3 = \{\{x_1, x_2, x_3, x_4\}^3, x_5(\{x_1, x_2, x_3\}^2 \setminus \{x_3^2\})\}.\ Therefore:$$

$$Shad(J_3) = \{\{x_1, x_2, x_3, x_4\}^4, x_5\{x_1, x_2, x_3, x_4\}^3, x_5^2(\{x_1, x_2, x_3\}^2 \setminus \{x_3^2\})\}.$$

We have $|Shad(J_3)| = 60$ and $|J_4| = |S_4| - |(S/J)_4| = 70 - 5 = 65$. So we need to add 5 new generators at $Shad(J_3)$ to get J_4. If we add a monomial which is divisible by x_5^3 we obtain a contradiction from the fact that the map $(S/J)_1 \overset{\cdot x_5^3}{\to} (S/J)_4$ is no longer injective. Therefore, we must add: $x_3^2x_5^2, x_1x_4x_5^2, x_2x_4x_5^2, x_3x_4x_5^2, x_4^2x_5^2$, and so

$$J_4 = \{\{x_1, x_2, x_3, x_4\}^4, x_5\{x_1, x_2, x_3, x_4\}^3, x_5^2\{x_1, x_2, x_3, x_4\}^2\}.$$

$$So\ \ Shad(J_4) = \{\{x_1, x_2, x_3, x_4\}^5, \cdots, x_5^3\{x_1, x_2, x_3, x_4\}^2\}.$$

We have $|J_5| - |Shad(J_4)| = 4$, so we must add 4 new generators at $Shad(J_4)$ to get J_5. Since J is strongly stable, these new generators are: $x_5^4x_1, x_5^4x_2, x_5^4x_3, x_5^4x_4$. Therefore $J_5 = \{\{x_1, \dots, x_5\}^5 \setminus \{x_5^5\}\}$. Finally, we must add x_5^6 to $Shad(J_5)$ in order to obtain J_6. We proved the following proposition, with the help of [6, Theorem 1.2] and Theorem 2.4.5.

Proposition 2.5.2. *If $I = (f_1, f_2, \dots, f_5) \subset K[x_1, \dots, x_5]$ is an ideal generated by a generic (regular) sequence of homogeneous polynomials of degree 2 or if $f_1, f_2, \dots, f_5$ is a regular sequence of homogeneous polynomials of degree 2 with $f_i \in K[x_i, \dots, x_5]$ for $i = 1, \dots, 5$ then $J = Gin(I)$ the generic initial ideal of I with respect to the revlex order is:*

$$J = (x_1^2,\ x_1x_2,\ x_2^2,\ x_1x_3,\ x_2x_3,\ x_3^3,\ x_3^2x_4,\ x_1x_4^2,\ x_2x_4^2,\ x_3x_4^2,\ x_4^3,$$
$$x_3^2x_5^2,\ x_1x_4x_5^2,\ x_2x_4x_5^2,\ x_3x_4x_5^2,\ x_4^2x_5^2,\ x_5^4x_1,\ x_5^4x_2,\ x_5^4x_3,\ x_5^4x_4,\ x_5^6)$$

The case $n = 4$, $d = 3$. We suppose that S/I has (SLP), so x_4 is a strong Lefschetz element for S/J. Also, we suppose that J_3 is revlex.

We have $H(S/J,t) = (1+t+t^2)^4 = (1+2t+3t^2+2t3+t^4)^2 =$

$$= 1 + 4t + 10t^2 + 16t^3 + 19t^4 + 16t^5 + 10t^6 + 4t^7 + t^8.$$

Since $|J_3| = 4$ and J_3 is revlex, it follows that $J_3 = \{x_1, x_2\}^3$. Therefore, we have $Shad(J_3) = \{\{x_1,x_2\}^4, \{x_1,x_2\}^3\{x_3,x_4\}\}$. Since $|J_4| - |Shad(J_3)| = 4$, we must add 4 new generators to $Shad(J_3)$ to obtain J_4. Since x_4 is a strong Lefschetz element for S/J we cannot add any monomial of the form $x_4 \cdot m$, where $m \neq 0$ in J_3. Therefore, since J is strongly stable, we have to choose 3 monomials from the list $x_3^2\{x_1,x_2\}^2, x_3^3\{x_1,x_2\}, x_3^4$. There are two different chooses: either we add (I) $x_3^2\{x_1,x_2\}^2$, either (II) $x_3^2x_1\{x_1,x_2,x_3\}$.

In the case (I), we get $J_4 = \{\{x_1,x_2\}^4, \{x_1,x_2\}^3\{x_3,x_4\}, x_3^2\{x_1,x_2\}^2\}$, so

$$Shad(J_4) = \{\{x_1,x_2\}^5, \{x_1,x_2\}^4\{x_3,x_4\}, \{x_1,x_2\}^3\{x_3,x_4\}^2, x_3^2\{x_3,x_4\}\{x_1,x_2\}^2\}.$$

Since $|J_5| - |Shad(J_4)| = 40 - 34 = 6$, we need to add 6 new generators at $Shad(J_4)$ to get J_5. Since x_4 is a strong Lefschetz element for S/J we cannot add any monomial of the form $x_4^2 m$, where m is a nonzero monomial in J_3. So, we must add: $x_3^4\{x_1,x_2,x_3\}, x_4x_3^3\{x_1,x_2,x_3\}$. Thus $J_5 = \{\{x_1,x_2,x_3\}^5, x_4\{x_1,x_2,x_3\}^4, x_4^2\{x_1,x_2\}^3\}$.

In the case (II), we have $J_4 = \{\{x_1,x_2\}^4, \{x_1,x_2\}^3\{x_3,x_4\}, x_1x_3^2\{x_1,x_2,x_3\}\}$, so $Shad(J_4)$ is the set $\{\{x_1,x_2\}^5, \{x_1,x_2\}^4\{x_3,x_4\}, \{x_1,x_2\}^3\{x_3,x_4\}^2, x_3^2x_1\{x_3,x_4\}\{x_1,x_2\}, x_3^3x_1\{x_3,x_4\}\}$. Since $|J_5| - |Shad(J_4)| = 40 - 34 = 6$, we must add 6 new generators at $Shad(J_4)$ to get J_5. Since x_4 is a strong-Lefschetz element for S/J, we cannot add any monomial of the form $x_4^2 m$, where $m \neq 0$ in J_3. So, we must add: $x_3^3x_2^2, x_3^4x_2, x_3^5, x_4x_3^2x_2^2, x_4x_3^3x_2, x_4x_3^4$. Thus

$$J_5 = \{\{x_1,x_2,x_3\}^5, x_4\{x_1,x_2,x_3\}^4, x_4^2\{x_1,x_2\}^3\},$$

the same as in the case (I). Thus, in both cases (I) and (II), we get:

$$Shad(J_5) = \{\{x_1,x_2,x_3\}^6, x_4\{x_1,x_2,x_3\}^5, x_4^2\{x_1,x_2,x_3\}^4, x_4^3\{x_1,x_2\}^3\}.$$

Since $|Shad(J_5)| = |S_6| - 16$ and $|J_6| = |S_6| - 10$, we must add 6 new generators to $Shad(J_5)$ in order to obtain J_6. Since x_4 is a strong-Lefschetz element for S/J, these new generators are not divisible by x_4^4. So, we add $x_4^3x_3\{x_1,x_2\}^2, x_4^3x_3^2\{x_1,x_2\}, x_4^3x_3^3$ and thus,

$$J_6 = \{\{x_1,x_2,x_3\}^6, x_4\{x_1,x_2,x_3\}^5, x_4^2\{x_1,x_2,x_3\}^4, x_4^3\{x_1,x_2,x_3\}^3\}. \textit{ So}$$

$$Shad(J_6) = \{\{x_1,x_2,x_3\}^7, x_4\{x_1,x_2,x_3\}^6, \ldots, x_4^4\{x_1,x_2,x_3\}^3\}.$$

$|S_7| - |Shad(J_6)| = 6 + 4 = 10$ and $|S_7| - |J_7| = 4$, so we must add 6 new generators at $Shad(J_6)$ to get J_7. Using the same argument, these new generators must be $x_4^5\{x_1,x_2,x_3\}^2$ and therefore $J_7 = \{\{x_1,x_2,x_3\}^7, x_4\{x_1,x_2,x_3\}^6, \ldots, x_4^5\{x_1,x_2,x_3\}^2\}$. We get

$$Shad(J_7) = \{\{x_1,x_2,x_3\}^8, x_4\{x_1,x_2,x_3\}^7, \ldots, x_4^6\{x_1,x_2,x_3\}^2\}.$$

Since $|S_8| - |Shad(J_7)| = 4$ and $|S_8| - |J_8| = 1$, we must add 3 new generators at $Shad(J_7)$ in order to get J_8. Since x_4 is strong-Lefschetz, these new generators are $x_4^7\{x_1,x_2,x_3\}$, so $J_8 = \{x_1,x_2,x_3,x_4\}^8 \setminus \{x_4^8\}$. Finally, we must add x_4^9 to $Shad(J_8)$ in order to obtain J_9. We proved the following proposition, with the help of [6, Theorem 1.2] and Theorem 2.4.5.

Proposition 2.5.3. *If $I = (f_1, f_2, f_3, f_4) \subset K[x_1, x_2, x_3, x_4]$ is an ideal generated by a generic (regular) sequence of homogeneous polynomials of degree 3 or if f_1, f_2, f_3, f_4 is a regular sequence of homogeneous polynomials of degree 3 with $f_i \in k[x_i, \ldots, x_4]$, for $i = 1, \ldots, 4$, then $J = Gin(I)$ the generic initial ideal of I with respect to the revlex order has one of the following forms:*

$$(I) \qquad J = (\{x_1, x_2\}^3,\ x_3^2\{x_1, x_2\}^2,\ x_3^4\{x_1, x_2, x_3\},\ x_4x_3^3\{x_1, x_2, x_3\},$$

$$x_4^3x_3\{x_1, x_2\}^2, x_4^3x_3^2\{x_1, x_2\}, x_4^3x_3^3,\ x_4^5\{x_1, x_2, x_3\}^2,\ x_4^7\{x_1, x_2, x_3\},\ x_4^9)$$

$$(II)\ J = (\{x_1, x_2\}^3,\ x_3^2x_1\{x_1, x_2, x_3\},\ x_3^3x_2^2,\ x_3^4x_2,\ x_3^5,\ x_4x_3^2x_2^2,\ x_4x_3^3x_2,\ x_4x_3^4,$$

$$x_4^3x_3\{x_1, x_2\}^2, x_4^3x_3^2\{x_1, x_2\}, x_4^3x_3^3,\ x_4^5\{x_1, x_2, x_3\}^2,\ x_4^7\{x_1, x_2, x_3\},\ x_4^9)$$

Remark 2.5.4. *It seems Conca-Herzog-Hibi noticed in [15], page 838, that, if f_1, f_2, f_3, f_4 is a generic sequence of homogeneous polynomials of degree 3 then the generic initial ideal J has the form (I), and $J = Gin(x_1^3, x_2^3, x_3^3, x_4^3)$ has the form (II).*

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